On Hölder calmness of solution mappings in parametric equilibrium problems

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Abstract We consider parametric equilibrium problems in metric spaces. Sufficient conditions for the Hölder calmness of solutions are established. We also study the Hölder well-posedness for equilibrium problems in metric spaces.

Keywords Equilibrium problem · calmness · well-posedness

1 Introduction

Optimization is one of the most fertile areas of mathematics. Its conclusions and recommendations play a very important role in both theoretical and applied mathematics. Equilibrium problems were first considered in Blum and Oettli (1994) and since then have been studied by many researchers all over the world. The equilibrium problem model incorporates many other important problems in optimization and other areas such as: variational inequalities, fixed point problems, complementarity, etc. There have been many studies of existence of solu-

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tions to equilibrium problems (see Sadequi and Alizadeh (2011), Bazán (2001), Bianchi and Schaible (1996), Hai and Khanh (2007a, 2007b), Hai et al. (2009)) and their stability, e.g., semi-continuity in the sense of Berge and Hausdorff (see Anh and Khanh (2004, 2007a, 2007b, 2010), Huang et al. (2006), Khanh and Luu (2007)) or Hölder (Lipschitzian) continuity (see Anh and Khanh (2006, 2008b, 2009), Bianchi and Pini (2003), Li et al. (2009), Li and Li (2011a, 2011b), Mansour and Riahi (2005)).

This paper extends Anh et al. (2012) and studies $(l.\alpha)$-*Hölder calmness* of solutions to parametric equilibrium problems. When $\alpha = 1$, this is a kind of calmness property which is in general stronger than the property of the same name usually used in variational analysis. Calmness property of multi-valued mappings has been examined by many authors (see Cánovas et al. (2009), Chuong et al. (2011), Henrion et al. (2002), Ioffe and Outrata (2008), Levy (2000), Ng and Zheng (2009)) in which subdifferentials and coderivatives play the main role. As applications we investigate conditions for Hölder calmness of solutions to optimization problems and well-posedness in the Hölder sense. The last subject is intimately related to the stability property and plays a very important role in studying optimization and variational problems.

The structure of the paper is as follows. Section 2 presents the equilibrium problem model and materials used in the rest of this paper. We establish in Section 3 a sufficient condition for the Hölder calmness of the mapping of solutions to parametric equilibrium problems. The Hölder well-posedness of equilibrium problems is studied in Section 4.

Throughout the paper, if not explicitly stated otherwise, X, Λ, M are metric spaces and $\mathbb{R}$ is the set of all real numbers while $\mathbb{R}_+$ is the set of all positive numbers. We use $d(\cdot,\cdot)$ for all metrics.

2 Preliminaries

For $K \subseteq X$ and a function $f : X \times X \to \mathbb{R}$ we consider the equilibrium problem

(EP) Find $\overline{x} \in K$ such that $f(\overline{x}, y) \geq 0$ for all $y \in K$.

The set of solutions to this problem is denoted by S. We consider also several extensions of this concept.

The constraint set K can be perturbed by a parameter $\lambda \in \Lambda$, $K : \Lambda \rightrightarrows X$, i.e., for each $\lambda \in \Lambda$, we can consider the problem

$(EP)_\lambda$ Find $\overline{x} \in K(\lambda)$ such that $f(\overline{x}, y) \geq 0$ for all $y \in K(\lambda)$.

We denote by $S(\lambda)$ the solution set of $(EP)_\lambda$.

The approximate version of this problem can be of interest: for each $\lambda \in \Lambda$ and $\varepsilon > 0$,

$(\widetilde{EP})_{\varepsilon,\lambda}$ Find $\overline{x} \in K(\lambda)$ such that $f(\overline{x}, y) + \varepsilon \geq 0$ for all $y \in K(\lambda)$.

We denote by $\widetilde{S}(\varepsilon, \lambda)$ the solution set of $(\widetilde{EP})_{\varepsilon,\lambda}$.

Problem $(EP)_\lambda$ can also be extended by allowing f to depend on another parameter: $f : X \times X \times M \to \mathbb{R}$, i.e., for each $(\lambda, \mu) \in \Lambda \times M$,

$(EP)_{\lambda,\mu}$ Find $\overline{x} \in K(\lambda)$ such that $f(\overline{x}, y, \mu) \geq 0$ for all $y \in K(\lambda)$.

The set of solutions to problem $(EP)_{\lambda,\mu}$ is denoted by $S(\lambda, \mu)$. This problem is going to be our main object of investigation in this paper.

Definition 2.1 For a function $f : X \to \mathbb{R}$ and positive numbers l, α,

(i) f is $(l.\alpha)$-Hölder continuous at $\overline{x}$ if there exists a neighborhood U of $\overline{x}$ such that

$$|f(x_1) - f(x_2)| \leq l d^\alpha(x_1, x_2) \text{ for all } x_1, x_2 \in U;$$

(ii) f is $(l.\alpha)$-Hölder calm at $\overline{x}$ if there exists a neighborhood U of $\overline{x}$ such that

$$|f(x) - f(\overline{x})| \leq l d^{\alpha}(x, \overline{x}) \text{ for all } x \in U.$$

From this definition, it is obvious that both properties are stronger than continuity in the usual sense while Hölder continuity is stronger than Hölder calmness.

To define similar properties for multi-values mappings we recall the following. For subsets A, B of X and a point $a \in X$,

$$d(a, B) = \inf_{b \in B} d(a, b);$$

$$H^*(A, B) = \sup_{a \in A} d(a, B);$$

$$H(A, B) = \max\{H^*(A, B), H^*(B, A)\};$$

$$\rho(A, B) = \sup_{a \in A, b \in B} d(a, b).$$

Note that H and ρ can take infinite values (if A or B is unbounded). It is also obvious that $H(A, B) \leq \rho(A, B)$ for any subsets A and B, and the inequality can be strict.

Example 2.1 Consider the subsets $A = [0; 1]$ and $B = (2; 4)$. We have

$$d(1, B) = \inf_{b \in B} d(1, b) = 1;$$

$$H^*(A, B) = \sup_{a \in A} d(a, B) = 2;$$

$$H^*(B, A) = \sup_{b \in B} d(b, A) = 3;$$

$$H(A, B) = \max\{H^*(A, B), H^*(B, A)\} = 3;$$

$$\rho(A, B) = \sup_{a \in A, b \in B} d(a, b) = 4.$$

$\rho(A, B) < \infty$ if and only if both A and B are nonempty and bounded. In the last example, if we take $B = (2; \infty)$ instead of $(2; 4)$, then

$$H^*(B, A) = H(A, B) = \rho(A, B) = \infty.$$

Definition 2.2 For a multi-valued mapping $K : \Lambda \rightrightarrows X$ and positive numbers l, α,

(i) K is $(l.\alpha)$-Hölder continuous at $\overline{\lambda}$ if there exists a neighborhood U of $\overline{\lambda}$ such that

$$H(K(\lambda_1), K(\lambda_2)) \leq l d^{\alpha}(\lambda_1, \lambda_2) \text{ for all } \lambda_1, \lambda_2 \in U;$$

(ii) K is $(l.\alpha)$-Hölder calm at $\overline{\lambda}$ if there exists a neighborhood U of $\overline{\lambda}$ such that

$$H(K(\lambda), K(\overline{\lambda})) \leq l d^{\alpha}(\lambda, \overline{\lambda}) \text{ for all } \lambda \in U. \tag{1}$$

Remark. The calmness in the above definition (when $\alpha = 1$) is a stronger property than the one usually considered in variational analysis. The latter corresponds to replacing H in (1) by H^* (see e.g. Rockafellar and Wets (1998)). Respectively, (l, α)-calmness is stronger than "calmness $[\alpha]$" in Kummer (2009).

Stronger properties of Hölder continuity/calmness with respect to ρ can also be of interest in applications.

We next define *uniform Hölder calmness* as the natural counterpart of relative Hölder continuity in Anh and Khanh (2007b).

Definition 2.3 For positive numbers m, β, θ, a function $f : X \times X \times M \to \mathbb{R}$ is $(m.\beta)$-Hölder calm at $\overline{\mu}$, θ-uniformly over a subset $S \subseteq X$ if there exists a neighborhood V of $\overline{\mu}$ such that

$$|f(x, y, \overline{\mu}) - f(x, y, \mu)| \leq m d^{\beta}(\overline{\mu}, \mu) d^{\theta}(x, y), \forall \mu \in V, \forall x, y \in S, x \neq y.$$

If $\theta = 0$, we say that f is $(m.\beta)$-Hölder calm in at $\overline{\mu}$, uniformly over S.

We next define several monotonicity properties playing a crucial role in examining the Hölder calmness of the solution mapping of the equilibrium problems $(EP)_{\lambda,\mu}$.

Definition 2.4 For a function $f : X \times X \to \mathbb{R}$, positive numbers h, β and a subset $S \subset X$,

(M_1) For all $x, y \in S, x \neq y$, it holds

$$f(x,y) + f(y,x) + hd^\beta(x,y) \leq 0; \tag{2}$$

(M_2) For all $x, y \in S$, it holds

$$hd^\beta(x,y) \leq d(f(x,y), \mathbb{R}_+) + d(f(y,x), \mathbb{R}_+); \tag{3}$$

(M_3) For all $x, y \in S, x \neq y$, it holds $[f(x,y) \geq 0 \Rightarrow f(y,x) + hd^\beta(x,y) \leq 0]$;

(M_4) For all $x, y \in S, x \neq y$, it holds $[f(x,y) < 0 \Rightarrow f(y,x) \geq 0]$.

The next proposition gives the relationships between these monotonicity properties.

Proposition 2.1

(i) $(M_1) \Rightarrow (M_2) \Rightarrow (M_3)$;

(ii) $[(M_3)\ \&\ (M_4)] \Rightarrow (M_2)$.

Proof. The following simple observation is used in the proof:

$$d(a, \mathbb{R}_+) = \max\{-a, 0\} \geq -a.$$

$(M_1) \Rightarrow (M_2)$. If (2) holds for some $x \neq y$, then

$$hd^\beta(x,y) \leq -f(x,y) - f(y,x) \leq d(f(x,y), \mathbb{R}_+) + d(f(y,x), \mathbb{R}_+),$$

i.e., (3) holds. When $x = y$, (3) holds automatically.

$(M_2) \Rightarrow (M_3)$. If (3) holds for some $x \neq y$ and $f(x, y) \geq 0$, then $d(f(x, y), \mathbb{R}_+) = 0$ and (3) takes the form

$$hd^\beta(x, y) \leq d(f(y, x), \mathbb{R}_+).$$

It follows from the last inequality that $d(f(y, x), \mathbb{R}_+) > 0$ and consequently $d(f(y, x), \mathbb{R}_+) = -f(y, x)$. Hence, (M_3) holds true.

$[(M_3)$ & $(M_4)] \Rightarrow (M_2)$. Let (M_3) and (M_4) hold true. We only need to prove (3) when $x \neq y$. If $f(x, y) \geq 0$, then $d(f(x, y), \mathbb{R}_+) = 0$ and (M_3) implies

$$0 < hd^\beta(x, y) \leq -f(y, x) = d(f(y, x), \mathbb{R}_+).$$

Hence, (3) is true. If $f(x, y) < 0$, then (M_4) implies $f(y, x) \geq 0$, and we can apply (M_3) again to show that

$$0 < hd^\beta(x, y) \leq -f(x, y) = d(f(x, y), \mathbb{R}_+).$$

Taking into account that $d(f(y, x), \mathbb{R}_+) = 0$, we conclude that (3) is true in this case too. $\square$

We now give examples showing that implications in Proposition 2.1 can be strict.

Example 2.2 The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x, y) = x - y$ satisfies (M_2) with $h = \beta = 1$. Indeed,

$$d(f(x, y), \mathbb{R}_+) + d(f(y, x), \mathbb{R}_+) = d(x - y, \mathbb{R}_+) + d(y - x, \mathbb{R}_+) = |x - y| = d(x, y).$$

At the same time, $f(x, y) + f(y, x) = 0$ and (2) is violated for any $x \neq y$. f does not satisfy (M_1). It is also obvious that f satisfies both (M_3) and (M_4).

Example 2.3 The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x, y) = -\frac{1}{4}(|x| + |y|)$ satisfies (M_3) with $h = \beta = 1$ as $f(x, y) \geq 0$ if and only if $x = y = 0$, it does not satisfy (M_2). Indeed, for any $y = -x \neq 0$, we have

$$d(f(x, y), \mathbb{R}_+) + d(f(y, x), \mathbb{R}_+) = \frac{1}{2}(|x| + |y|) = |x| < 2|x| = d(x, y).$$

We can see that the combination of (M_3) and (M_4) implies (M_2), but they are not equivalent by considering the function $f(x, y) = -(|x| + |y|)$. This function satisfies (M_2) with $h = \beta = 1$, but breaks (M_4).

Remark. Properties (M_1), (M_3) and (M_4) were considered in Anh and Khanh (2006, 2007b, 2008a) where they were called Hölder strong monotonicity, Hölder strong pseudo-monotonicity and quasi-monotonicity respectively.

3 The Hölder calmness of the mapping of solutions

The next theorem improving Theorem 2.1 in Anh at al. (2012) gives a sufficient condition for the Hölder calmness of the solution mapping of the problem $(EP)_{\lambda,\mu}$. We always assume that solution sets $S(\lambda, \mu)$ are nonempty for all (λ, μ) in a neighborhood of the considered point $(\overline{\lambda}, \overline{\mu})$.

Theorem 3.1 *Consider equilibrium problem $(EP)_{\lambda,\mu}$ and suppose the following conditions hold.*

(i) *There exist neighborhoods $U(\overline{\lambda})$ of $\overline{\lambda}$ and $V(\overline{\mu})$ of $\overline{\mu}$ and positive numbers n_1, δ_1 and θ such that f is $(n_1.\delta_1)$-Hölder calm at $\overline{\mu}$, θ-uniformly over $K(U(\overline{\lambda}))$.*

(ii) *There exist positive numbers n_2 and δ_2 such that, for all $x \in K(U(\overline{\lambda}))$ and $\mu \in V(\overline{\mu})$, the function $f(x, \cdot, \mu)$ is $(n_2.\delta_2)$-Hölder continuous on $K(U(\overline{\lambda}))$.*

(iii) $f(\cdot,\cdot,\overline{\mu})$ *satisfies condition* (M_2) *with constants* $h>0$ *and* $\beta>\theta$.

(iv) K *is* $(l.\alpha)$*-Hölder calm at* $\overline{\lambda}$ *with some positive* l *and* α.

Then solutions to $(EP)_{\lambda,\mu}$ *satisfy the condition of Hölder calmness with respect to* ρ*: there exist constants* $k_1,k_2>0$ *such that*

$$\rho\big(S(\overline{\lambda},\overline{\mu}),S(\lambda,\mu)\big)\le k_1 d^{\alpha\delta_2/\beta}(\overline{\lambda},\lambda)+k_2 d^{\delta_1/(\beta-\theta)}(\overline{\mu},\mu),$$

for all (λ,μ) *in a neighborhood of* $(\overline{\lambda},\overline{\mu})$.

Proof. Take $\lambda\in U(\overline{\lambda})$ and $\mu\in V(\overline{\mu})$.

Step 1. We prove that for each $x(\lambda,\overline{\mu})\in S(\lambda,\overline{\mu})$ and $x(\lambda,\mu)\in S(\lambda,\mu)$,

$$d_1:=d\big(x(\lambda,\overline{\mu}),x(\lambda,\mu)\big)\le\Big(\frac{n_1}{h}\Big)^{1/(\beta-\theta)}d^{\delta_1/(\beta-\theta)}(\overline{\mu},\mu). \tag{4}$$

Suppose $x(\lambda,\overline{\mu})\neq x(\lambda,\mu)$, because both $x(\lambda,\overline{\mu})$ and $x(\lambda,\mu)$ belong to $K(\lambda)$ and are solutions of $(EP)_{\lambda,\mu}$,

$$f\big(x(\lambda,\overline{\mu}),x(\lambda,\mu),\overline{\mu}\big)\ge 0; \tag{5}$$

$$f\big(x(\lambda,\mu),x(\lambda,\overline{\mu}),\mu\big)\ge 0. \tag{6}$$

At the same time, (iii) implies

$$d\Big(f\big(x(\lambda,\overline{\mu}),x(\lambda,\mu),\overline{\mu}\big),\mathbb{R}_+\Big)+d\Big(f\big(x(\lambda,\mu),x(\lambda,\overline{\mu}),\overline{\mu}\big),\mathbb{R}_+\Big)\ge hd_1^{\beta}.$$

Combining this inequality with (5) and (6), we get

$$d\Big(f\big(x(\lambda,\mu),x(\lambda,\overline{\mu}),\overline{\mu}\big),f\big(x(\lambda,\mu),x(\lambda,\overline{\mu}),\mu\big)\Big)\ge hd_1^{\beta}.$$

Because f is $(n_1.\delta_1)$-Hölder calm at $\overline{\mu}$, θ-uniformly over $K(U(\overline{\lambda}))$ by (i), the above relationship implies

$$n_1 d_1^{\theta} d^{\delta_1}(\overline{\mu},\mu)\ge hd_1^{\beta}.$$

This is equivalent to $d_1^{\beta-\theta} \leq \frac{n_1}{h} d^{\delta_1}(\overline{\mu}, \mu)$ from which we get (4) proved.

Step 2. We prove that for each $x(\overline{\lambda}, \overline{\mu}) \in S(\overline{\lambda}, \overline{\mu})$ and $x(\lambda, \overline{\mu}) \in S(\lambda, \overline{\mu})$,

$$d_2 := d\big(x(\overline{\lambda}, \overline{\mu}), x(\lambda, \overline{\mu})\big) \leq \Big(\frac{2n_2 l^{\delta_2}}{h}\Big)^{1/\beta} d^{\alpha\delta_2/\beta}(\overline{\lambda}, \lambda). \tag{7}$$

Suppose $x(\overline{\lambda}, \overline{\mu}) \neq x(\lambda, \overline{\mu})$. (iv) implies that there exist $\overline{x} \in K(\overline{\lambda})$ and $x \in K(\lambda)$ such that

$$d\big(x(\overline{\lambda}, \overline{\mu}), x\big) \leq l d^{\alpha}(\overline{\lambda}, \lambda); \tag{8}$$

$$d\big(x(\lambda, \overline{\mu}), \overline{x}\big) \leq l d^{\alpha}(\overline{\lambda}, \lambda). \tag{9}$$

We get from the definition of $(EP)_{\lambda,\mu}$,

$$f\big(x(\overline{\lambda}, \overline{\mu}), \overline{x}, \overline{\mu}\big) \geq 0; \tag{10}$$

$$f\big(x(\lambda, \overline{\mu}), x, \overline{\mu}\big) \geq 0. \tag{11}$$

At the same time, (iii) implies

$$d\Big(f\big(x(\overline{\lambda}, \overline{\mu}), x(\lambda, \overline{\mu}), \overline{\mu}\big), \mathbb{R}_+\Big) + d\Big(f\big(x(\lambda, \overline{\mu}), x(\overline{\lambda}, \overline{\mu}), \overline{\mu}\big), \mathbb{R}_+\Big) \geq h d_2^{\beta}.$$

Combining this inequality with (10) and (11), we get

$$\begin{aligned} d\Big(f\big(x(\overline{\lambda}, \overline{\mu}), x(\lambda, \overline{\mu}), \overline{\mu}\big), f\big(x(\overline{\lambda}, \overline{\mu}), \overline{x}, \overline{\mu}\big)\Big) \\ + d\Big(f\big(x(\lambda, \overline{\mu}), x(\overline{\lambda}, \overline{\mu}), \overline{\mu}\big), f\big(x(\lambda, \overline{\mu}), x, \overline{\mu}\big)\Big) \geq h d_2^{\beta}. \end{aligned}$$

Because f is $(n_2\delta_2)$-Hölder continuous with respect to the second component in $K(U(\overline{\lambda}))$ by (ii),

$$n_2 d^{\delta_2}\big(x(\lambda, \overline{\mu})), \overline{x}\big) + n_2 d^{\delta_2}\big(x(\overline{\lambda}, \overline{\mu}), x\big) \geq h d_2^{\beta}.$$

We combine this with (8) and (9) and get

$$n_2 l^{\delta_2} d^{\alpha\delta_2}(\overline{\lambda}, \lambda) + n_2 l^{\delta_2} d^{\alpha\delta_2}(\overline{\lambda}, \lambda) \geq h d_2^{\beta},$$

or equivalently $d_2^\beta \leq \frac{2n_2 l^{\delta_2}}{h} d^{\alpha\delta_2}(\overline{\lambda}, \lambda)$. We have (7) proved.

Step 3. For all $x(\overline{\lambda}, \overline{\mu}) \in S(\overline{\lambda}, \overline{\mu})$ and $x(\lambda, \mu) \in S(\lambda, \mu)$, we always have

$$d\big(x(\overline{\lambda}, \overline{\mu}), x(\lambda, \mu)\big) \leq d_1 + d_2.$$

From (4) and (7), by taking $k_1 = \Big(\frac{2n_2 l^{\delta_2}}{h}\Big)^{1/\beta}$ and $k_2 = \Big(\frac{n_1}{h}\Big)^{1/(\beta-\theta)}$, we get

$$\rho\big(S(\overline{\lambda}, \overline{\mu}), S(\lambda, \mu)\big) \leq k_1 d^{\alpha\delta_2/\beta}(\overline{\lambda}, \lambda) + k_2 d^{\delta_1/(\beta-\theta)}(\overline{\mu}, \mu).$$

Therefore, Theorem 3.1 has been proved. □

Normally, to receive a property of solution mappings, the problem's hypotheses are also required at the level corresponding to that property. We can see from the preceding theorem that all the hypotheses are related to Hölder continuity and Hölder calmness.

The next example indicates the essential role of assumption (iii) in Theorem 3.1.

Example 3.1 Take $X = M = \Lambda = \mathbb{R}$, $K(\lambda) = [-1; \infty)$ for all $\lambda \in \mathbb{R}$. For each $\mu \in \mathbb{R}$, consider the function f defined by $f(x, y, \mu) = x + y + |\mu|$. Take $\overline{\mu} = 0$.

We have $|f(x, y, \mu) - f(x, y, \overline{\mu})| = |\mu|$ for all $x, y \in [-1; \infty)$, so f is (1.1)-Hölder calm at $\overline{\mu}$, uniformly over $[-1; \infty)$. At the same time, $|f(x, y, \mu) - f(x, z, \mu)| = |y - z|$ for all $y, z \in [-1; \infty)$; this means that $f(x, \cdot, \mu)$ is (1.1)-Hölder continuous on $[-1; \infty)$. Hence, conditions (i) and (ii) are fulfilled.

Condition (iv) is also true straightforwardly. However, we have

$$S(0) = [1; \infty),\ S(\mu) = [1 - |\mu|; \infty),$$

so $\rho\big(S(\mu), S(0)\big) = \infty$ for any μ.

Therefore, the solution mapping S is not Hölder calm at $\overline{\mu} = 0$. The reason here is that f breaks condition (M_2). Indeed,

$$d\big(f(1,2,0),\mathbb{R}_+\big)+d\big(f(2,1,0),\mathbb{R}_+\big)=0<h|1-2|^{\beta}=h,\forall h,\beta>0.$$

Condition (M_2) in Theorem 3.1 is indispensable.

Remark 3.1 It follows from Proposition 2.1 that the conclusion of Theorem 3.1 remains true if condition (iii) is replaced by either condition (M_1) or conditions (M_3) and (M_4).

The next proposition aims to illustrate application of Theorem 3.1. For each $(\lambda,\mu)\in\Lambda\times M$, we consider the minimization problem

(MP) Minimize $f(x,\mu)$ subject to $x\in K(\lambda)$,

where $f:X\times M\to\mathbb{R}$ and $K:\Lambda\rightrightarrows X$. We denote $S(\lambda,\mu)=\big\{\overline{x}\in K(\lambda): f(\overline{x},\mu)=\min_{x\in K(\lambda)}\{f(x,\mu)\}\big\}$ and assume that $S(\lambda,\mu)\neq\emptyset$ for all (λ,μ) near the considered point $(\overline{\lambda},\overline{\mu})$.

Proposition 3.2 *Consider (MP) and suppose the following conditions hold.*

(i) *There exist neighborhoods $V(\overline{\mu})$ of $\overline{\mu}$ and $U(\overline{\lambda})$ of $\overline{\lambda}$ and numbers $n_1>0$ and $\delta_1>0$ such that f is $(n_1.\delta_1)$-Hölder calm at $\overline{\mu}$, uniformly over $K(U(\overline{\lambda}))$, i.e.,*

$$|f(x,\mu)-f(x,\overline{\mu})|\leq n_1d^{\delta_1}(\mu,\overline{\mu})$$

for all $x\in K(U(\overline{\lambda}))$ and $\mu\in V(\overline{\mu})$.

(ii) *There exist numbers $n_2>0$ and $\delta_2>0$ such that f is $(n_2.\delta_2)$-Hölder continuous in x on $K(U(\overline{\lambda}))$, uniformly over $\mu\in V(\overline{\mu})$, i.e.,*

$$|f(x,\mu)-f(y,\mu)|\leq n_2d^{\delta_2}(x,y) \tag{12}$$

for all $\mu\in V(\overline{\mu})$ and $x,y\in K(U(\overline{\lambda}))$, and (12) holds as an equality when $\mu=\overline{\mu}$.

(iii) *K is $(l.\alpha)$-Hölder calm at $\overline{\lambda}$ with some $l > 0$ and $\alpha > 0$.*

Then the mapping S is Hölder calm with respect to ρ, i.e., there exist constants $k_1, k_2 > 0$ such that

$$\rho\big(S(\overline{\lambda}, \overline{\mu}), S(\lambda, \mu)\big) \leq k_1 d^{\alpha}(\overline{\lambda}, \lambda) + k_2 d(\overline{\mu}, \mu), \tag{13}$$

for all (λ, μ) in a neighborhood of $(\overline{\lambda}, \overline{\mu})$.

Proof. We define the function g as follows

$$g : X \times X \times M \to \mathbb{R},\ \ g(x, y, \mu) = f(y, \mu) - f(x, \mu).$$

We observe that $\overline{x} \in S(\lambda, \mu)$ if and only if $\overline{x} \in K(\lambda)$ and $g(\overline{x}, y, \mu) \geq 0, \forall y \in K(\lambda)$. So to prove the proposition, it suffices to check that g satisfies the conditions of Theorem 3.1.

We first check condition (i). For every $\mu \in V(\overline{\mu})$ and $x, y \in K(U(\overline{\lambda}))$ we have

$$|g(x, y, \mu) - g(x, y, \overline{\mu})| = |f(y, \mu) - f(x, \mu) - f(y, \overline{\mu}) + f(x, \overline{\mu})|$$

$$\leq |f(x, \mu) - f(x, \overline{\mu})| + |f(y, \mu) - f(y, \overline{\mu})| \leq 2n_1 d^{\delta_1}(\mu, \overline{\mu}).$$

This means that g is $(2n_1.\delta_1)$-Hölder calm at $\overline{\mu}$, uniformly over $K(U(\overline{\lambda}))$.

We have at the same time

$$|g(x, y, \mu) - g(x, z, \mu)| = |f(y, \mu) - f(z, \mu)| \leq n_2 d^{\delta_2}(y, z),$$

i.e., g is $(n_2.\delta_2)$-Hölder continuous with respect to the second component. So conditions (i) and (ii) in Theorem 3.1 are fulfilled.

We now check condition (iii) in Theorem 3.1. For all $x, y \in K(U(\overline{\lambda}))$, we have

$$d\big(g(x, y, \overline{\mu}), \mathbb{R}_+\big) + d\big(g(y, x, \overline{\mu}), \mathbb{R}_+\big)$$

$$= d\big(f(y, \overline{\mu}) - f(x, \overline{\mu}), \mathbb{R}_+\big) + d\big(f(x, \overline{\mu}) - f(y, \overline{\mu}), \mathbb{R}_+\big)$$

$$= |f(x, \overline{\mu}) - f(y, \overline{\mu})| = n_2 d^{\delta_2}(x, y).$$

So g satisfies condition (M_2), and (iii) in Theorem 3.1 is fulfilled. Therefore, it follows from Theorem 3.1 that (13) holds true with some $k_1, k_2 > 0$. $\square$

4 The Hölder well-posedness of equilibrium problems

We will denote by $(\mathcal{EP})$ the family of problems $\{(EP)_\lambda : \lambda \in \Lambda\}$ and extend the concept of Lipschitzian well-posedness for optimization problems introduced in Bednarczuk (2007) to equilibrium problems.

Definition 4.1 $(\mathcal{EP})$ is Hölder well-posed at $\overline{\lambda}$ if $\widetilde{S}(0, \overline{\lambda})$ is nonempty and $\widetilde{S}$ is Hölder calm at $(0, \overline{\lambda})$.

The next theorem (a version of this result can be found in Anh et al. (2012)) gives a sufficient condition for the Hölder well-posedness of $(\mathcal{EP})$.

Theorem 4.1 *Assume $S(\overline{\lambda}) \neq \emptyset$ and the following conditions hold.*

(i) *There exist a neighborhood $U(\overline{\lambda})$ of $\overline{\lambda}$ and numbers $n_2 > 0$ and $\delta_2 > 0$ such that for all $x \in K(U(\overline{\lambda}))$, the function $f(x, \cdot)$ is $(n_2.\delta_2)$-Hölder continuous on $K(U(\overline{\lambda}))$.*

(ii) *f satisfies condition (M_2) with some positive constants h, β.*

(iii) *K is $(l.\alpha)$-Hölder calm at $\overline{\lambda}$ with some positive constants l, α.*

Then $(\mathcal{EP})$ is Hölder well-posed at $\overline{\lambda}$.

Proof. We define a function $g : X \times X \times [0; \infty) \to \mathbb{R}$ as follows

$$g(x, y, \varepsilon) = f(x, y) + \varepsilon.$$

To prove the Theorem, it suffices to check that g satisfies the conditions of Theorem 3.1.

Take any neighborhood V of 0 in $[0;\infty)$. Then for all $\varepsilon \in V$ and $x, y \in K(U(\overline{\lambda}))$,

$$|g(x,y,\varepsilon) - g(x,y,0)| = |f(x,y) + \varepsilon - f(x,y) - 0| = \varepsilon.$$

So g is (1.1)-Hölder calm at 0, uniformly over $K(U(\overline{\lambda}))$.

We have at the same time

$$|g(x,y,\varepsilon) - g(x,z,\varepsilon)| = |f(x,y) - f(x,z)| \leq n_2 d^{\delta_2}(y,z),$$

or g is $(n_2.\delta_2)$-Hölder continuous with respect to the second component on $K(U(\overline{\lambda}))$. Conditions (i) and (ii) of Theorem 3.1 are fulfilled.

We now check condition (iii) of Theorem 3.1. For all $x, y \in K(U(\overline{\lambda}))$, we get

$$d\big(g(x,y,0),\mathbb{R}_+\big) + d\big(g(y,x,0),\mathbb{R}_+\big) = d\big(f(x,y),\mathbb{R}_+\big) + d\big(f(y,x),\mathbb{R}_+\big) \geq h d^{\beta}(x,y).$$

This means that g satisfies condition (iii) of Theorem 3.1 and we have all its hypotheses satisfied. Therefore, the mapping of solutions to $(\mathcal{EP})$ is Hölder calm which combined with Definition 4.1 gives the conclusion of the theorem. $\square$

5 Conclusion

Assuming Hölder calmness and Hölder continuity in Hausdorff distance, we have established the Hölder calm property of the solution mapping with respect to ρ. This obviously implies the Hölder calm property in Hausdorff distance. We have established a sufficient condition for the Hölder well-posedness of equilibrium problems. These may be extended to many other classes of problems.

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