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Extensions of metric regularity

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The paper is devoted to some extensions of the metric regularity property for mappings between metric or Banach spaces. Several new concepts are investigated in a unified manner: uniform metric regularity, metric regularity along a subspace, metric multi-regularity for mappings into product spaces (when each component is perturbed independently), as well as their Lipschitz-like counterparts. The properties are characterized in terms of certain derivative-like constants. Regularity criteria are established based on a set-valued extension of a nonlocal version of Lyusternik-Graves theorem due to Milyutin.

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1. Introduction

The property of metric regularity has proved to be one of the fundamental concepts of the contemporary variational analysis, playing an extremely important role both in theory and its numerous applications to investigating *well-posedness* or *sensitivity* of generalized equations, variational inequalities, optimization problems, etc. Being well established and recognized this concept continues its expansion into new areas of mathematical analysis (see the recent monographs [18, 24] and survey papers [2, 13]).

Recall that a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces is said to be *metrically regular* near $(x^\circ, y^\circ) \in \operatorname{gph} F$ if there exist $\kappa > 0$ and $\delta > 0$ such that

$$d(x, F^{-1}(y)) \le \kappa d(y, F(x)), \qquad \forall x \in B_\delta(x^\circ),\ y \in B_\delta(y^\circ). \tag{1}$$

Here F^{-1} denotes the *inverse* mapping: $F^{-1}(y) = \{u \in X :\ F(u) \ni y\}$.

In the case of a single-valued mapping f between Banach spaces, strictly differentiable at x°, the main regularity criterion is provided by the famous Lyusternik-Graves theorem (see Corollary 2.7), which basically reduces the problem of regularity of f to that of its linear approximation $\nabla f(x^\circ)$, the criterion being the surjectivity of $\nabla f(x^\circ)$ (see [8, 13]).

In the general case, some extensions of Lyusternik-Graves theorem have been developed based on the same idea: to reduce the problem of regularity of F to that

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of another mapping close to F in a certain sense. This idea can be traced back to the original papers by Lyusternik and Graves (see the extensions of Lyusternik-Graves theorem in [6, 8, 15]).

Despite its great generality, Lyusternik-Graves theorem in its standard form as well as its traditional extensions does not cover some important cases. One of the examples is given by systems of generalized equations [5], when conditions are needed guaranteeing the existence of error bound type estimates when the variables in all equations are perturbed independently.

Conditions of this type do not follow directly from the standard (local) Lyusternik-Graves theorem. Fortunately, once again classical results happen to be richer than they are usually cited. It was noticed by Milyutin in early seventies (see e.g. the survey paper [6]) that the original proof of Lyusternik's result was applicable to the "nonlocal" metric regularity, when the estimate of type (1) held on a prescribed set. Although well known to the variational analysis community, the importance of this feature of Milyutin's extension of Lyusternik-Graves theorem seems to be still underestimated. At the same time this is exactly the property needed to prove the "parametric" version of Lyusternik-Graves theorem (see Theorem 3.3) and deduce the necessary estimates for systems of generalized equations (see Theorem 6.1). It can also be useful for other applications (see [4]).

In this paper which continues [5] and does not intend to surprise specialists on variational analysis, we investigate in a unified manner several metric regularity properties for set-valued mappings:

(i) conventional local (defined by (1)) and nonlocal *metric regularity* (Section 2);
(ii) *uniform metric regularity* for mappings depending on a parameter (Section 3);
(iii) *metric regularity along a subspace* (Section 4);
(iv) *metric multi-regularity* for mappings into product spaces (Section 5)

and formulate corresponding Lyusternik-Graves type regularity criteria.

In Section 2 we formulate and prove a set-valued extension of the extended Lyusternik-Graves theorem due to Milyutin [6]. The proof of Theorem 2.1 follows the original ideas of Lyusternik and Milyutin. It can be extended to cover the case of *restrictive metric regularity* [18].

Properties (ii)–(iv) generalize local metric regularity. On the other hand, the following implications hold true:

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv),$$

meaning that each of the properties (ii)–(iv) can be treated as a particular case of the previous one. It is important to emphasize here that uniform metric regularity (ii) is implied by nonlocal metric regularity; a local version is not sufficient.

It is not surprising that each of the above regularity properties has the corresponding Lipschitz-like (pseudo-Lipschitz) counterpart, which is used when formulating the regularity criteria for a perturbed mapping. So starting with Section 2 devoted to the conventional metric regularity, we always formulate pairs of properties (regularity and Lipschitzness). The only exception is the metric multi-regularity property considered in Section 5. We show in Proposition 5.4 that the corresponding Lipschitz-like property coincides with the conventional one.

To characterize properties (i)–(iv) as well as their Lipschitz-like counterparts we define certain derivative-like constants and establish certain relationships between them. The regularity criteria follow as corollaries of the corresponding relationships.

Property (iv) for mappings into product spaces corresponds to the case when each component of the mapping is perturbed independently. This allows to define in Section 6 *metric multi-regularity* of systems of generalized equations and to formulate some "error bound" type regularity criteria, which were formerly deduced in [5] from the "parametric" Lyusternik-Graves theorem. The multi-regularity can be interpreted as a stability concept permitting different reactions of "players" in a complex optimization problem to different perturbations.

Mainly standard notations are used throughout the paper. A closed ball of radius ρ centered at x in a metric space is denoted by $B_\rho(x)$. We write B_ρ if $x = 0$. Distance is denoted by $d(\cdot,\cdot)$. The same notation is used for point-to-set distances: $d(x,\Omega) = \inf_{\omega\in\Omega} d(x,\omega)$. When considering products of metric (normed) spaces, we always assume that they are equipped with the maximum-type distance (norm): $d((x_1,y_1),(x_2,y_2)) = \max\{d(x_1,x_2), d(y_1,y_2)\}$ ($\|(x,y)\| = \max\{\|x\|,\|y\|\}$). The notation $F : X \rightrightarrows Y$ is used for set-valued mappings (the term "set-valued" is usually omitted) between metric spaces as opposed to single-valued functions $f : X \to Y$.

2. Metric regularity and Lyusternik-Graves theorem

Recall that a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces is said to be *metrically regular* if there exists $\kappa > 0$ such that

$$d(x, F^{-1}(y)) \le \kappa d(y, F(x)), \qquad \forall x \in X,\ y \in Y. \tag{2}$$

This important property is usually characterized by the *modulus* (or *norm*) *of metric regularity* $\operatorname{reg} F$ which is defined as the infimum of all κ for which (2) holds: F is metrically regular if and only if $\operatorname{reg} F < \infty$. Another convenient tool for characterizing metric regularity is the following (possibly infinite) derivative-like constant:

$$r[F] = \inf_{x\in X,\, y\in Y} \left[\frac{d(y,F(x))}{d(x,F^{-1}(y))}\right]_\infty. \tag{3}$$

Here $[\cdot/\cdot]_\infty$ is the "extended" division operation, which differs from the usual one in the additional rule $[0/0]_\infty = +\infty$. This allows one not to worry about the points (x,y) with $y \in F(x)$ when evaluating the infimum in (3).

Clearly,

$$r[F] = \frac{1}{\operatorname{reg} F},$$

and the criterion of metric regularity can be written as $r[F] > 0$.

Another important property closely related to (2) and in a sense opposite to it, is the so called *pseudo-Lipschitz* property: there exists $\kappa > 0$ such that

$$d(y, F(x)) \le \kappa d(x, x'), \qquad \forall x, x' \in X,\ \forall y \in F(x'). \tag{4}$$

It can be characterized by the following derivative-like constant:

$$l[F] = \sup_{x,x'\in X,\, y\in F(x')} \left[\frac{d(y,F(x))}{d(x,x')}\right]_0 = \sup_{x\in X,\, y\in Y} \left[\frac{d(y,F(x))}{d(x,F^{-1}(y))}\right]_0, \tag{5}$$

which obviously coincides with the infimum of all κ for which (4) holds. Here $[\cdot/\cdot]_0$ stands for another "extended" division operation with the additional rule $[0/0]_0 = 0$. Pseudo-Lipschitzness of F is equivalent to the inequality $l[F] < \infty$.

One can easily notice that

$$l[F] = \sup_{x,x' \in X} \left[\frac{h(F(x), F(x'))}{d(x, x')} \right]_0 ,$$

where $h(F(x), F(x'))$ is the *Hausdorff distance* between the sets $F(x)$ and $F(x')$.

It follows immediately from the definition, that

$$l[F] = \frac{1}{r[F^{-1}]}.$$

For a single-valued function $f : X \to Y$, constant (5) becomes the usual *Lipschitz modulus*:

$$l[f] = \sup_{x,x' \in X} \left[\frac{d(f(x), f(x'))}{d(x, x')} \right]_0 . \tag{6}$$

Thus, f is Lipschitz continuous if and only if $l[f] < \infty$.

Using (3) and (5), one can formulate the following extension of the extended Lyusternik-Graves theorem due to Milyutin [6].

Theorem 2.1: *Let $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Y$ be mappings from a complete metric space into a normed linear space. Suppose that* $\operatorname{gph} F$ *is closed in $X \times Y$ and the following condition holds true:*

(C) If $(x_n, y_n) \in \operatorname{gph} G$, $x_n \to \hat{x}$ and y_n is a fundamental (Cauchy) sequence, then y_n converges to some $\hat{y} \in G(\hat{x})$.

Then

$$r[F + G] \geq r[F] - l[G].$$

Obviously the conclusion of the above theorem is meaningful only if $l[G] < r[F]$. Then it guarantees metric regularity of the perturbed mapping $F + G$ when F is metrically regular and the perturbation G is sufficiently "small". If $l[G] < \infty$ then condition (C) is ensured, for instance, by any of the following conditions:

- G is single-valued;
- G is compact-valued;
- Y is complete and $\operatorname{gph} G$ is closed.

The proof of Theorem 2.1 presented below is a modification of the one given in [4] for single-valued mappings and follows the original ideas of Lyusternik and Milyutin.

Proof: If $r[F] \leq l[G]$, the assertion is trivial, so we assume that $r[F] > l[G]$. Let real numbers a and b satisfy $r[F] > a > b > l[G]$. Denote $\Phi = F + G$. We are going to show that $r[\Phi] \geq a - b$.

Take arbitrary $(x_0, y_0) \in \operatorname{gph} \Phi$ and $\hat{y} \in Y$. Denote $r = \|y_0 - \hat{y}\|$. We need to find $\hat{x} \in \Phi^{-1}(\hat{y})$ such that

$$d(x_0, \hat{x}) \leq r/(a - b).$$

If $r = 0$, that is $\hat{y} = y_0$, one can take $\hat{x} = x_0$. From now on we assume that $r \neq 0$. The point $\hat{x}$ will be obtained as the limit of a sequence $\{x_n\}$ generated by a special iteration process.

By the definition of Φ one has $y_0 = v_0 + w_0$ where $v_0 \in F(x_0)$ and $w_0 \in G(x_0)$. Consider the point $v_1 = \hat{y} - w_0 \in Y$. Obviously $\|v_1 - v_0\| = \|y_0 - \hat{y}\| = r$. It follows from the definition (3) that

$$a < \frac{\|v_1 - v_0\|}{d(x_0, F^{-1}(v_1))}.$$

Consequently, there exists $x_1 \in F^{-1}(v_1)$ such that

$$d(x_0, x_1) \leq \|v_1 - v_0\|/a = r/a.$$

It follows from (5) that there exists $w_1 \in G(x_1)$ such that

$$\|w_0 - w_1\| \leq bd(x_0, x_1) \leq (b/a)r.$$

Now, repeating the same procedure again starting with x_1, v_1, w_1 instead of x_0, v_0, w_0, we construct a triplet x_2, v_2, w_2 satisfying

$$v_2 \in F(x_2), \quad w_2 \in G(x_2),$$
$$v_2 = \hat{y} - w_1,$$
$$d(x_1, x_2) \leq \|v_2 - v_1\|/a = \|w_0 - w_1\|/a \leq (b/a)r/a,$$
$$\|w_1 - w_2\| \leq bd(x_1, x_2) \leq (b/a)^2 r.$$

Continuing this process, we obtain a sequence of triplets x_n, v_n, w_n such that

$$v_n \in F(x_n), \quad w_n \in G(x_n), \tag{7}$$
$$v_n = \hat{y} - w_{n-1}, \tag{8}$$
$$d(x_n, x_{n+1}) \leq (b/a)^n r/a, \tag{9}$$
$$\|w_{n-1} - w_n\| \leq (b/a)^n r. \tag{10}$$

Inequalities (9) and (10) imply that $\{x_n\}$ and $\{w_n\}$ are fundamental sequences, and since X is complete and condition (C) holds true, these sequences converge to some $\hat{x}$ and $\hat{w}$ respectively, with $\hat{w} \in G(\hat{x})$. Then (8) implies that $v_n \to \hat{v} = \hat{y} - \hat{w}$. Due to (7) and the closedness of the graph of F we have $\hat{v} \in F(\hat{x})$. Consequently, $\hat{y} = \hat{v} + \hat{w} \in \Phi(\hat{x})$. Besides, (9) implies the estimate

$$d(x_0, \hat{x}) \leq \sum_{n=0}^{\infty} d(x_n, x_{n+1}) \leq \left(1 + (b/a) + (b/a)^2 + \ldots\right)(r/a) \leq r/(a - b).$$

Thus, $r[\Phi] \geq a - b$. Since a and b can be taken arbitrarily close to $r[F]$ and $l[G]$ respectively, the proof is completed. □

Remark 1 : A different approach to investigating set-valued perturbations of a regular mapping based on another way of measuring the closeness of multifunctions is pursued in [14, 19].

Remark 2: Given a set $V \subset X \times Y$, similarly to [13], one can define metric regularity and pseudo-Lipschitzness of F on V and formulate the corresponding generalization of Theorem 2.1. Along these lines it is possible to cover the case of *restrictive metric regularity* (see [18, Remark 1.61]).

Remark 3: Theorem 2.1 remains true if Y is a linear metric space with the metric invariant with respect to translations.

In the rest of the paper we limit ourselves to the case when $G = g$ is a single-valued mapping.

Applying Theorem 2.1 to the sum of the set-valued mapping $F + g$ and the function $-g$, one immediately obtains an upper estimate for $r[F + g]$.

Corollary 2.2: *Under the conditions of Theorem 2.1,*

$$r[F] - l[g] \leq r[F + g] \leq r[F] + l[g].$$

Remark 4: Formally Corollary 2.2 follows from Theorem 2.1 only when together with the graph of F, the graph of $F+g$ is also closed, which is the case, for instance, if g is continuous. However, if g is discontinuous, then $l[g] = \infty$, and the conclusion of Corollary 2.2 holds true automatically.

In the rest of the section we are interested in the local properties of the mapping F near a given point $(x^\circ, y^\circ) \in \operatorname{gph} F$ (the graph of F):

- *metric regularity*: there exist $\kappa > 0$ and $\delta > 0$ such that

$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x)), \qquad \forall x \in B_\delta(x^\circ),\ y \in B_\delta(y^\circ);$$

- *pseudo-Lipschitz* (*Aubin's Lipschitz-like*) property: there exist $\kappa > 0$ and $\delta > 0$ such that

$$d(y, F(x)) \leq \kappa d(x, x'), \qquad \forall x, x' \in B_\delta(x^\circ),\ y \in F(x') \cap B_\delta(y^\circ).$$

Both properties can be considered as particular cases of the corresponding nonlocal ones. If any of them holds for some $\kappa > 0$ and $\delta > 0$, then the restriction of F considered as a mapping between metric spaces $B_\delta(x^\circ)$ and $B_\delta(y^\circ)$ satisfies the corresponding nonlocal property.

The constants (3), (5) applied to the mapping $F : B_\delta(x^\circ) \rightrightarrows B_\delta(y^\circ)$ take the following form:

$$r_\delta[F](x^\circ, y^\circ) = \inf_{x \in B_\delta(x^\circ),\, y \in B_\delta(y^\circ)} \left[\frac{d(y, F(x))}{d(x, F^{-1}(y))} \right]_\infty, \tag{11}$$

$$l_\delta[F](x^\circ, y^\circ) = \sup_{\substack{x, x' \in B_\delta(x^\circ) \\ y \in F(x') \cap B_\delta(y^\circ)}} \left[\frac{d(y, F(x))}{d(x, x')} \right]_0. \tag{12}$$

Clearly F is metrically regular (pseudo-Lipschitz) near (x°, y°) if and only if $r_\delta[F](x^\circ, y^\circ) > 0$ ($l_\delta[F](x^\circ, y^\circ) < \infty$) for some $\delta > 0$.

If $f : X \to Y$ is a single-valued function, then $y^\circ = f(x^\circ)$, and we shall write $r_\delta[f](x^\circ)$ and $l_\delta[f](x^\circ)$ instead of $r_\delta[f](x^\circ, y^\circ)$ and $l_\delta[f](x^\circ, y^\circ)$, respectively. In this case, (12) can be rewritten as

$$l_\delta[f](x^\circ) = \sup_{\substack{x, x' \in B_\delta(x^\circ) \\ f(x') \in B_\delta(f(x^\circ))}} \left[\frac{d(f(x), f(x'))}{d(x, x')} \right]_0. \tag{13}$$

Alongside (13), it can be of interest to consider a slightly simplified constant:

$$l'_\delta[f](x^\circ) = \sup_{x,x' \in B_\delta(x^\circ)} \left[\frac{d(f(x), f(x'))}{d(x,x')} \right]_0. \tag{14}$$

Proposition 2.3:

(i) $l_\delta[f](x^\circ) \le l'_\delta[f](x^\circ)$ *for any* $\delta > 0$*;*
(ii) $\lim\limits_{\delta\to+0} l_\delta[f](x^\circ) = \lim\limits_{\delta\to+0} l'_\delta[f](x^\circ)$.

Proof: The first assertion follows directly from the definitions (13), (14). It immediately implies the inequality $\lim\limits_{\delta\to+0} l_\delta[f](x^\circ) \le \lim\limits_{\delta\to+0} l'_\delta[f](x^\circ)$. Consider arbitrary sequences $\{x_k\}$ and $\{x'_k\}$ converging to x°. To prove the opposite inequality we need to show that

$$\limsup_{k\to\infty} \frac{d(f(x_k), f(x'_k))}{d(x_k, x'_k)} \le \lim_{\delta\to+0} l_\delta[f](x^\circ). \tag{15}$$

Assume that $f(x'_k) \to f(x^\circ)$. Then for any $\delta > 0$ and all sufficiently large k one has $x_k, x'_k \in B_\delta(x^\circ)$, $f(x'_k) \in B_\delta(f(x^\circ))$, and consequently

$$\frac{d(f(x_k), f(x'_k))}{d(x_k, x'_k)} \le l_\delta[f](x^\circ).$$

This implies (15).

If $f(x'_k) \not\to f(x^\circ)$, taking subsequences if necessary, we can assume that there exists $\alpha > 0$ such that $d(f(x'_k), f(x^\circ)) > \alpha$ for all sufficiently large k. Then

$$\lim_{\delta\to+0} l_\delta[f](x^\circ) \ge \lim_{k\to\infty} \frac{d(f(x'_k), f(x^\circ))}{d(x'_k, x^\circ)} = \infty,$$

and (15) holds true automatically. □

Remark 5: Inequality (i) in Proposition 2.3 can be strict. Take $f(x) = 2x$ for $x \le 2$ and $f(x) = 5x - 6$ for $x > 2$. Then $l_3[f](0) = (f(3) - f(1.5))/(3 - 1.5) = 4$ while $l'_3[f](0) = 5$.

Corollary 2.2 applied to the appropriate restrictions of F, g and $F + g$ produces the following result.

Theorem 2.4: *Let* $F : X \rightrightarrows Y$ *and* $g : X \to Y$ *be mappings from a complete metric space into a normed linear space. Suppose that* $\operatorname{gph} F$ *is closed in* $X \times Y$ *and* $(x^\circ, y^\circ) \in \operatorname{gph} F$*. Then for any* $\delta > 0$,

$$r_\delta[F](x^\circ, y^\circ) - l_\delta[g](x^\circ) \le r_\delta[F+g](x^\circ, y^\circ + g(x^\circ)) \le r_\delta[F](x^\circ, y^\circ) + l_\delta[g](x^\circ). \tag{16}$$

Remark 6: The assumption that the graph of F is closed in Theorem 2.4 can be obviously weakened: it is sufficient to assume that it is locally closed near (x°, y°): if $\operatorname{gph} F \cap B_{\delta_0}(x^\circ, y^\circ)$ is closed for some $\delta_0 > 0$ then (16) holds for any positive $\delta < \delta_0$.

Remark 7: Note the "nonlocal" character of Theorem 2.4: if $l_\delta[g](x^\circ) < r_\delta[F](x^\circ, y^\circ)$, then the first inequality in (16) guarantees metric regularity of $F + g$ in the neighborhood of $(x^\circ, y^\circ + g(x^\circ))$ of the same radius δ. This is important for some applications (see Theorem 3.1 below).

The limiting versions of (11), (12), (13) (or (14); due to Proposition 2.3 (ii), (13) and (14) produce the same limiting constant) can also be useful:

$$r[F](x^\circ, y^\circ) = \lim_{\delta\to 0} r_\delta[F](x^\circ, y^\circ),$$
$$l[F](x^\circ, y^\circ) = \lim_{\delta\to 0} l_\delta[F](x^\circ, y^\circ), \quad l[f](x^\circ) = \lim_{\delta\to 0} l_\delta[f](x^\circ).$$

Moreover, F is metrically regular (pseudo-Lipschitz) near (x°, y°) if and only if $r[F](x^\circ, y^\circ) > 0$ ($l[F](x^\circ, y^\circ) < \infty$). The constants provide quantitative characterizations of the corresponding properties.

The explicit representations of the constants follow from their definitions:

$$r[F](x^\circ, y^\circ) = \liminf_{x\to x^\circ,\, y\to y^\circ} \left[\frac{d(y, F(x))}{d(x, F^{-1}(y))}\right]_\infty, \tag{17}$$

$$l[F](x^\circ, y^\circ) = \limsup_{\substack{x,x'\to x^\circ,\, y\to y^\circ\\ y\in F(x')}} \left[\frac{d(y, F(x))}{d(x, x')}\right]_0, \tag{18}$$

$$l[f](x^\circ) = \limsup_{x,x'\to x^\circ} \left[\frac{d(f(x), f(x'))}{d(x, x')}\right]_0. \tag{19}$$

Constant (17) is also known as the *rate* or *modulus of surjection* or *covering* (see [13, 18]) and is often denoted $\operatorname{sur} F(x^\circ|y^\circ)$ or $\operatorname{cov} F(x^\circ|y^\circ)$. If $f : X \to Y$ is single-valued, we write $r[f](x^\circ)$ instead of $r[f](x^\circ, f(x^\circ))$.

Remark 8: If $f : X \to Y$ is continuous and $r[f](x^\circ) > 0$, then Y is complete [4].

Constant (18) is exactly the *Lipschitz modulus* $\operatorname{lip} F(x^\circ|y^\circ)$ [24]. It coincides with (19) in the case of a single-valued function $f : X \to Y$. If $l[g](x^\circ) = 0$ then g is *strictly stationary* at x° [8, 10]. In this case, for any set-valued mapping F the mapping $F + g$ is a *strict first-order approximation* [12, 18] to F (and vice versa) at x°.

If a single-valued function $f : X \to Y$ between normed spaces is *strictly differentiable* at x° (see [18, 24]), that is there exists a linear and continuous mapping $\nabla f(x^\circ) : X \to Y$ (the *strict derivative* of f at x°) such that

$$\lim_{\substack{x,x'\to x^\circ\\ x'\neq x}} \frac{f(x') - f(x) - \nabla f(x^\circ)(x' - x)}{\|x' - x\|} = 0,$$

then $l(f - \nabla f(x^\circ)) = 0$.

Taking $\delta \to 0$ in (16) one immediately obtains the following result.

Corollary 2.5: *Under the conditions of Theorem 2.4,*

$$r[F](x^\circ, y^\circ) - l[g](x^\circ) \le r[F + g](x^\circ, y^\circ + g(x^\circ)) \le r[F](x^\circ, y^\circ) + l[g](x^\circ). \tag{20}$$

The first inequality in (20) is sometimes referred to (in the Banach space setting) as the *extended Lyusternik-Graves theorem* [12]. It can be viewed as an estimate of the *radius of metric regularity* of a perturbed mapping with respect to single-valued Lipschitz continuous perturbations (see [11, 12]).

Remark 9: The conclusion of Corollary 2.5 is in general weaker than that of Theorem 2.4: if $l[g](x^\circ) < r[F](x^\circ, y^\circ)$, then the first inequality in (20) guarantees metric regularity of $F + g$, but does not provide any estimate of the radius of the neighborhood of $(x^\circ, y^\circ + g(x^\circ))$ where this property holds.

An important application of Corollary 2.5 is when $l[g](x^\circ) = 0$.

Corollary 2.6: *Let $l[g](x^\circ) = 0$ in the conditions of Corollary 2.5. Then*

$$r[F + g](x^\circ, y^\circ + g(x^\circ)) = r[F](x^\circ, y^\circ).$$

This means that a perturbation of a mapping by a strictly stationary function does not affect its metric regularity. This result allows one to reduce examining metric regularity of a complicated mapping to that of a simpler (usually linear) one. Taking into account that for a linear and continuous mapping between Banach spaces its metric regularity does not depend on the point and is equivalent to the surjectivity (Banach open mapping theorem – see [1, 8, 13]), Corollary 2.6 implies the following assertion.

Corollary 2.7: *Let a function $f : X \to Y$ between Banach spaces be strictly differentiable at x°. Then f is metrically regular near x° if and only if $\nabla f(x^\circ)$ is surjective, and indeed $r[f](x^\circ) = r[\nabla f(x^\circ)](0) = ||(\nabla f(x^\circ)^*)^{-1}||^{-1}$.*

Sometimes Corollary 2.7 is referred to as Lyusternik-Graves theorem (see [11]), as its sufficient part actually recaptures the classical version of the famous theorem.

Based on Corollary 2.6 it is possible to consider a more general case of "partial strict linearization" for the sum $f + F$ of a strictly differentiable function f and an arbitrary set-valued mapping F with closed graph (see [12]). It is also possible to consider mappings, which admit nonlinear (in particular, positively homogenous) approximations.

3. Uniform metric regularity

The statement of Corollary 2.6 (as well as Corollary 2.7) can be extended to the important for applications case of mappings depending on a parameter, the main tool being the same Theorem 2.4.

Consider a mapping $F : P \times X \rightrightarrows Y$, where X and Y are metric spaces and P is a topological space. Denote $F_p = F(p, \cdot) : X \rightrightarrows Y$. Let $(p^\circ, x^\circ, y^\circ) \in \operatorname{gph} F$.

In this section we examine the following local properties of F near $(p^\circ, x^\circ, y^\circ)$:

- *uniform metric regularity*: there exist $\kappa > 0$ and $\delta > 0$ such that

$$d(x, F_p^{-1}(y)) \le \kappa d(y, F(p, x)), \quad \forall x \in B_\delta(x^\circ),\ y \in B_\delta(y^\circ),\ p \in B_\delta(p^\circ).$$

- *uniform pseudo-Lipschitz* (*Aubin's Lipschitz-like*) property: there exist $\kappa > 0$ and $\delta > 0$ such that

$$d(y, F(p, x)) \le \kappa d(x, x'), \qquad \forall x, x' \in B_\delta(x^\circ),\ y \in F(p, x') \cap B_\delta(y^\circ),\ p \in B_\delta(p^\circ).$$

These properties generalize the corresponding ones considered in Section 2 and can be characterized by the following analogs of (11), (12) depending on some positive δ:

$$r_\delta[F](p^\circ;x^\circ,y^\circ) = \inf_{x\in B_\delta(x^\circ),\, y\in B_\delta(y^\circ),\, p\in B_\delta(p^\circ)} \left[\frac{d(y,F(p,x))}{d(x,F_p^{-1}(y))}\right]_\infty, \tag{21}$$

$$l_\delta[F](p^\circ;x^\circ,y^\circ) = \sup_{\substack{x,x'\in B_\delta(x^\circ),\, p\in B_\delta(p^\circ)\\ y\in F(p,x')\cap B_\delta(y^\circ)}} \left[\frac{d(y,F(p,x))}{d(x,x')}\right]_0 \tag{22}$$

and their limiting counterparts:

$$r[F](p^\circ;x^\circ,y^\circ) = \liminf_{x\to x^\circ,\, y\to y^\circ,\, p\to p^\circ} \left[\frac{d(y,F(p,x))}{d(x,F_p^{-1}(y))}\right]_\infty,$$

$$l[F](p^\circ;x^\circ,y^\circ) = \limsup_{\substack{x,x'\to x^\circ,\, y\to y^\circ,\, p\to p^\circ\\ y\in F(p,x')}} \left[\frac{d(y,F(p,x))}{d(x,x')}\right]_0.$$

The relationships between (11), (12) and (21), (22) respectively, are straightforward:

$$r_\delta[F](p^\circ;x^\circ,y^\circ) = \inf_{p\in B_\delta(p^\circ)} r_\delta[F_p](x^\circ,y^\circ), \tag{23}$$

$$l_\delta[F](p^\circ;x^\circ,y^\circ) = \sup_{p\in B_\delta(p^\circ)} l_\delta[F_p](x^\circ,y^\circ).$$

If $f: P\times X\to Y$ is single-valued we write $r_\delta[f](p^\circ;x^\circ)$, $l_\delta[f](p^\circ;x^\circ)$, $r[f](p^\circ;x^\circ)$ and $l[f](p^\circ;x^\circ)$. In this case,

$$l_\delta[f](p^\circ;x^\circ) = \sup_{\substack{x,x'\in B_\delta(x^\circ),\, p\in B_\delta(p^\circ)\\ f(p,x')\in B_\delta(f(p^\circ,x^\circ))}} \left[\frac{d(f(p,x),f(p,x'))}{d(x,x')}\right]_0, \tag{24}$$

$$l[f](p^\circ;x^\circ) = \limsup_{x,x'\to x^\circ,\, p\to p^\circ} \left[\frac{d(f(p,x),f(p,x'))}{d(x,x')}\right]_0. \tag{25}$$

Instead of (24), in many cases, it is possible to use a simplified constant

$$l'_\delta[f](p^\circ;x^\circ) = \sup_{x,x'\in B_\delta(x^\circ),\, p\in B_\delta(p^\circ)} \left[\frac{d(f(p,x),f(p,x'))}{d(x,x')}\right]_0.$$

Obviously,

$$l_\delta[f](p^\circ;x^\circ) = \sup_{p\in B_\delta(p^\circ)} l_\delta[f_p](x^\circ), \tag{26}$$

$$l'_\delta[f](p^\circ;x^\circ) = \sup_{p\in B_\delta(p^\circ)} l'_\delta[f_p](x^\circ).$$

The condition $l[f](p^\circ;x^\circ)<\infty$ (or $l'_\delta[f](p^\circ;x^\circ)<\infty$ for some $\delta>0$) means that f is locally *uniformly Lipschitz* [7, 9] (*equi-Lipschitz* [3]) in x at (p°,x°).

When X and Y are normed spaces, a function $f: P\times X\to Y$ is said to be *partially strictly differentiable* in x [20] (see [3]) at (p°,x°), if there exists a linear and continuous mapping $\nabla_x f(p^\circ, x^\circ) : X \to Y$ such that

$$\lim_{\substack{x,x' \to x^\circ,\, p \to p^\circ \\ x' \neq x}} \frac{f(p,x') - f(p,x) - \nabla_x f(p^\circ, x^\circ)(x' - x)}{\|x' - x\|} = 0.$$

Clearly this is a direct generalization of the notion of *strict differentiability* which corresponds to the case when f does not depend on p. The above condition is equivalent to the equality $l[f - \nabla_x f(p^\circ, x^\circ)](p^\circ; x^\circ) = 0$, which means that $\nabla_x f(p^\circ, x^\circ)$ *strictly approximates* f in x at (p°, x°) (see [23] where the term "strongly" is used). Note also that this property is stronger than the strict differentiability of the function $x \to f(p^\circ, x)$ at x°.

Theorem 2.4 and conditions (23), (26) lead to the following assertion.

Theorem 3.1 : *Let $F : P \times X \rightrightarrows Y$ and $g : P \times X \to Y$, where X is a complete metric space, Y is a normed space, and P is a topological space. Suppose that* $\operatorname{gph} F$ *is closed in $P \times X \times Y$ and $(p^\circ, x^\circ, y^\circ) \in \operatorname{gph} F$. Then for any $\delta > 0$,*

$$|r_\delta[F + g](p^\circ; x^\circ, y^\circ + g(x^\circ)) - r_\delta[F](p^\circ; x^\circ, y^\circ)| \le l_\delta[g](p^\circ; x^\circ).$$

The limiting statements follow immediately.

Corollary 3.2: *Under the conditions of Theorem 3.1*

$$|r[F + g](p^\circ; x^\circ, y^\circ + g(x^\circ)) - r[F](p^\circ; x^\circ, y^\circ)| \le l[g](p^\circ; x^\circ).$$

Corollary 3.3: *Let $l[g](p^\circ; x^\circ) = 0$ in the conditions of Corollary 3.2. Then*

$$r[F + g](p^\circ; x^\circ, y^\circ + g(p^\circ, x^\circ)) = r[F](p^\circ; x^\circ, y^\circ).$$

Remark 1 : Corollaries 3.2 and 3.3 cannot be deduced from the corresponding "local" Corollaries 2.5 and 2.6. One needs the full "nonlocal" Theorem 2.4.

Corollary 3.2 clearly implies the parametric version of Corollary 2.7. The next statement considers a slightly more general case. With the help of Corollary 3.3 one can reduce examining metric regularity of the sum $f + F$ of a partially strictly differentiable in x at (p°, x°) function $f : P \times X \to Y$ and an arbitrary set-valued mapping $F : X \rightrightarrows Y$ with closed graph to that of its partial strict linearization $h \to \nabla_x f(p^\circ, x^\circ)h + F(x^\circ + h)$.

Corollary 3.4: *Let $F : X \rightrightarrows Y$ and $f : P \times X \to Y$, where X is a Banach space, Y is a normed space, and P is a topological space. Suppose that* $\operatorname{gph} F$ *is closed in $X \times Y$, $(x^\circ, y^\circ) \in \operatorname{gph} F$ and f is partially strictly differentiable in x at (p°, x°). Then*

$$r[f + F](p^\circ; x^\circ, y^\circ + f(p^\circ, x^\circ)) = r[\nabla_x f(p^\circ, x^\circ) + \hat{F}](0, y^\circ),$$

where $\hat{F}(h) = F(x^\circ + h)$, $h \in X$.

Remark 2 : Corollary 3.4 follows from Corollary 3.3 due to the fact that the linear function $h \to \nabla_x f(p^\circ, x^\circ)h$ strictly approximates f in x at (p°, x°). Obviously one can formulate a more general assertion: the function f does not have to be partially strictly differentiable; it is sufficient to assume that it can be strictly approximated in x at (p°, x°) by another function g. This allows to recover the main statement of [9, Theorem 4.1] which is a generalization of the Robinson's landmark result in [22].

If in the conditions of Corollary 3.4 the mapping F is convex (gph F is convex in $X \times Y$), then $\nabla_x f(p^\circ, x^\circ) + \hat{F}$ is convex too, and the regularity criterion is provided by Robinson-Ursescu theorem (see [1, 11, 18]):

$$y^\circ \in \operatorname{int}\, [\nabla_x f(p^\circ, x^\circ) + \hat{F}](X)$$

(cf. [9, Theorem 3.3]).

The next assertion addresses a particular case, when $F(x) = Q$, if $x \in C$ and $F(x) = \emptyset$ otherwise, where Q is a closed convex cone in Y and C is a closed convex subset in X.

Corollary 3.5: *Let X and Y be Banach spaces and P – a topological space; let Q be a closed convex cone in Y and C – a closed convex subset in X; let a function $f : P \times X \to Y$ be partially strictly differentiable in x at (p°, x°) and $x^\circ \in C$, $-f(p^\circ, x^\circ) \in Q$. Consider the mapping $\Phi(p, x) = f(p, x) + Q$, if $x \in C$, and $\Phi(p, x) = \emptyset$ otherwise. Then Φ is uniformly metrically regular near $(p^\circ, 0, x^\circ)$ with respect to (x, y) if and only if*

$$0 \in \operatorname{int}\, \{f(p^\circ, x^\circ) + \nabla_x f(p^\circ, x^\circ)(C - x^\circ) + Q\}. \tag{27}$$

This is essentially Robinson theorem [21] (see also [3]). The role of the cone Q is to represent inequality and equality constrains in optimization problems. Condition (27) reduces to the surjectivity of $\nabla_x f(p^\circ, x^\circ)$ when $C = X$ and $Q = \{0\}$, that is there are equality constraints only.

4. Metric regularity along a subspace

Let $F : X \rightrightarrows Y$ be a mapping between normed linear spaces and let H be a (closed) subspace of X.

In this section we examine the following local properties of F near $(x^\circ, y^\circ) \in$ gph F:

- *metric regularity along H*: there exist $\kappa > 0$ and $\delta > 0$ such that

$$\inf_{h \in H} \{\|h\| : x + h \in F^{-1}(y)\} \le \kappa d(y, F(x)), \quad \forall x \in B_\delta(x^\circ),\ y \in B_\delta(y^\circ).$$

- *pseudo-Lipschitz* (*Aubin's Lipschitz-like*) property *along H*: there exist $\kappa > 0$ and $\delta > 0$ such that

$$d(y, F(x)) \le \kappa \|h\|, \qquad \forall x \in B_\delta(x^\circ),\ h \in H \cap B_\delta,\ y \in F(x + h) \cap B_\delta(y^\circ).$$

Obviously, if $H = X$ then these properties coincide with traditional respectively metric regularity and pseudo-Lipschitzness of F near (x°, y°).

In the definition of metric regularity along H, it is convenient to use the concept of *distance along H* defined for $x \in X$ and $M \subset X$ as

$$d_H(x, M) = \inf_{h \in H} \{\|h\| :\ x + h \in M\} = d(0, (M - x) \cap H).$$

Of course, it is not a real distance on X. For instance, $d_H(x_1, x_2) = \infty$ if $x_1 - x_2 \notin H$. In general, $d_H(x_1, x_2) \ge d(x_1, x_2)$ for any $x_1, x_2 \in X$, and the equality holds when $H = X$.

The above concepts can be characterized by the following constants:

$$r_{H,\delta}[F](x^\circ,y^\circ) = \inf_{x\in B_\delta(x^\circ),\, y\in B_\delta(y^\circ)} \left[\frac{d(y,F(x))}{d_H(x,F^{-1}(y))}\right]_\infty, \tag{28}$$

$$l_{H,\delta}[F](x^\circ,y^\circ) = \sup_{\substack{x\in B_\delta(x^\circ),\, h\in H\cap B_\delta \\ y\in F(x+h)\cap B_\delta(y^\circ)}} \left[\frac{d(y,F(x))}{\|h\|}\right]_0, \tag{29}$$

$$r_H[F](x^\circ,y^\circ) = \liminf_{x\to x^\circ,\, y\to y^\circ} \left[\frac{d(y,F(x))}{d_H(x,F^{-1}(y))}\right]_\infty, \tag{30}$$

$$l_H[F](x^\circ,y^\circ) = \limsup_{\substack{x\to x^\circ,\, h\xrightarrow{H}0,\, y\to y^\circ \\ y\in F(x+h)}} \left[\frac{d(y,F(x))}{\|h\|}\right]_0,$$

which reduce to (11), (12), (17), (18) when $H = X$. The notation $h \xrightarrow{H} 0$ in the last formula means that $h \in H$ approaches $0 \in H$.

Evidently,

$$r_{H,\delta}[F](x^\circ,y^\circ) \le r_\delta[F](x^\circ,y^\circ), \qquad l_{H,\delta}[F](x^\circ,y^\circ) \le l_\delta[F](x^\circ,y^\circ),$$
$$r_H[F](x^\circ,y^\circ) \le r[F](x^\circ,y^\circ), \qquad l_H[F](x^\circ,y^\circ) \le l[F](x^\circ,y^\circ),$$

and metric regularity of F along some subspace implies its standard metric regularity, while standard pseudo-Lipschitz property F implies its pseudo-Lipschitz property along any subspace.

We write $r_{H,\delta}[f](x^\circ)$, $l_{H,\delta}[f](x^\circ)$, $r_H[f](x^\circ)$ and $l_H[f](x^\circ)$ if $f : X \to Y$ is single-valued. The last constant can be expressed equivalently as

$$l_H[f](x^\circ) = \limsup_{x\to x^\circ\ h\xrightarrow{H}0} \frac{d(f(x+h),f(x))}{\|h\|}. \tag{31}$$

If $l_H[f](x^\circ) < \infty$ we say, that f is *Lipschitz continuous along* H near x°.

Similarly we say, that f is *strictly differentiable along* H at x°, if there exists a linear and continuous mapping $\nabla_H f(x^\circ) : H \to Y$ such that

$$\lim_{x\to x^\circ\ h\xrightarrow{H}0} \frac{f(x+h) - f(x) - \nabla_H f(x^\circ)h}{\|h\|} = 0.$$

The last equality is equivalent to the condition $l_H[f - \nabla_H f(x^\circ)](x^\circ) = 0$.

Standard Lipschitz continuity or strict differentiability imply the corresponding properties along any subspace.

All the defined above “along a subspace” properties as well as the corresponding constants can be treated in the framework of the “parametric” properties considered in Section 3.

For the mapping $F : X \rightrightarrows Y$, define another mapping $\Phi : X \times H \rightrightarrows Y$ by the formula

$$\Phi(x,h) = F(x+h), \quad x \in X,\ h \in H. \tag{32}$$

Then, for this mapping, X can be viewed as a parameter space and the definitions and results of the preceding section can be reformulated for this particular case, the point $h^\circ = 0$ being of special interest. The next Proposition shows that uniform metric regularity (uniform pseudo-Lipschitz property) of Φ near $(x^\circ, 0, y^\circ)$ is exactly metric regularity (pseudo-Lipschitz property) of F near (x°, y°) along H.

Proposition 4.1: *Let the mapping $\Phi : X \times H \rightrightarrows Y$ be defined by (32) and let $\delta > 0$. Then*

(i) $r_{H,\delta}[F](x^\circ, y^\circ) \ge r_\delta[\Phi](x^\circ; 0, y^\circ) \ge r_{H,2\delta}[F](x^\circ, y^\circ)$;
(ii) $l_{H,\delta}[F](x^\circ, y^\circ) \le l_\delta[\Phi](x^\circ; 0, y^\circ) \le l_{H,2\delta}[F](x^\circ, y^\circ)$;
(iii) $r_H[F](x^\circ, y^\circ) = r[\Phi](x^\circ; 0, y^\circ)$;
(iv) $l_H[F](x^\circ, y^\circ) = l[\Phi](x^\circ; 0, y^\circ)$.

Proof: Taking into account (32) and the obvious relations $F(x) = \Phi(x, 0)$ and

$$\Phi_x^{-1}(y) = (F^{-1}(y) - x) \cap H, \qquad d(h, \Phi_x^{-1}(y)) = d_H(x + h, F^{-1}(y)),$$

the first inequalities in (i) and (ii) follow from the definitions (21), (22), (28), and (29):

$$r_\delta[\Phi](x^\circ; 0, y^\circ) \le \inf_{x \in B_\delta(x^\circ),\, y \in B_\delta(y^\circ)} \left[\frac{d(y, \Phi(x, 0))}{d(0, \Phi_x^{-1}(y))} \right]_\infty = r_{H,\delta}[F](x^\circ, y^\circ),$$
$$l_\delta[\Phi](x^\circ; 0, y^\circ) \ge \sup_{\substack{x \in B_\delta(x^\circ),\, h \in H \cap B_\delta \\ y \in \Phi(x,h) \cap B_\delta(y^\circ)}} \left[\frac{d(y, \Phi(x, 0))}{\|h\|} \right]_0 = l_{H,\delta}[F](x^\circ, y^\circ).$$

On the other hand,

$$r_\delta[\Phi](x^\circ; 0, y^\circ) = \inf_{x \in B_\delta(x^\circ),\, y \in B_\delta(y^\circ),\, h \in B_\delta} \left[\frac{d(y, F(x + h))}{d_H(x + h, F^{-1}(y))} \right]_\infty \ge r_{H,2\delta}[F](x^\circ, y^\circ),$$
$$l_\delta[\Phi](x^\circ; 0, y^\circ) = \sup_{\substack{x \in B_\delta(x^\circ),\, h, h' \in B_\delta \\ y \in F(x+h) \cap B_\delta(y^\circ)}} \left[\frac{d(y, F(x + h'))}{\|h - h'\|} \right]_0 \le l_{H,2\delta}[F](x^\circ, y^\circ),$$

because $x + h \in B_{2\delta}(x^\circ)$ in the first of the above relations, and $x + h' \in B_{2\delta}(x^\circ)$ and $\|h - h'\| \le 2\delta$ in the second one. This proves the remaining inequalities in (i) and (ii).

The equalities (iii) and (iv) are corollaries of (i) and (ii). □

Similarly, in the case of a single-valued function $f : X \to Y$, its Lipschitz continuity near x° and strict differentiability at x° along H coincide with uniform Lipschitz continuity and partial strict differentiability in h at $(x^\circ, 0)$ of the function $\phi : X \times H \to Y$, given by $\phi(x, h) = f(x + h)$, $h \in H$, $x \in X$, and the constant (25) reduces to (31).

Application of Theorem 3.1 to the setting adopted in this section yields the following statement.

Theorem 4.2: *Let $F : X \rightrightarrows Y$ and $g : X \to Y$, where X is a Banach space and Y is a normed space. Suppose that* gph F *is closed in $X \times Y$ and $(x^\circ, y^\circ) \in$* gph F*. Then for any subspace $H \subset X$ and any $\delta > 0$,*

$$|r_{H,\delta}[F + g](x^\circ, y^\circ + g(x^\circ)) - r_{H,\delta}[F](x^\circ, y^\circ)| \le l_{H,\delta}[g](x^\circ).$$

A collection of corollaries follows as usually.

Corollary 4.3: *Under the conditions of Theorem 4.2*

$$|r_H[F+g](x^\circ, y^\circ + g(x^\circ)) - r_H[F](x^\circ, y^\circ)| \le l_H[g](x^\circ).$$

Corollary 4.4: *Let $l_H[g](x^\circ) = 0$ in the conditions of Corollary 4.3. Then*

$$r_H[F+g](x^\circ, y^\circ + g_H(x^\circ)) = r_H[F](x^\circ, y^\circ).$$

The next result is a realization of Corollary 3.4.

Theorem 4.5: *Let $f : X \to Y$ and $F : H \rightrightarrows Y$, where X is a Banach space, Y is a normed linear space, and H is a subspace of X. Suppose that $\operatorname{gph} F$ is closed in $H \times Y$, $(0, y^\circ) \in \operatorname{gph} F$, and f is strictly differentiable along the subspace H at x°. Then*

$$r[\phi + F](x^\circ; 0, y^\circ + f(x^\circ)) = r[\nabla_H f(x^\circ) + F](0, y^\circ), \tag{33}$$

where $\phi(x, h) = f(x + h)$, $h \in H$, $x \in X$.

Remark 1: Note that in the left hand side of (33) the function of two variables ϕ replaces the initial function f, and the "parametric" regularity constant is used, not the constant introduced in this section. This is because the sum $f + F$ is not defined on the whole of X.

A special case of Theorem 4.5 can be of interest for applications: when F is a constant mapping, that is $F(h) = Q$ for all $h \in H$, where Q is a closed subset (not necessarily a cone) of Y. In this case, the mapping $\phi + F : X \times H \rightrightarrows Y$ actually depends on the sum of its variables:

$$(\phi + F)(x, h) = (f + Q)(x + h), \quad h \in H,\ x \in X.$$

This makes it possible to replace the "parametric" regularity constant in the left hand side of (33) with the one defined by (30).

Corollary 4.6: *Let $f : X \to Y$ be a function from a Banach space into a normed linear space and Q be a closed subset of Y. Suppose that f is strictly differentiable along a subspace $H \subset X$ at x° and $y^\circ \in Q$. Then*

$$r_H[f + Q](x^\circ, y^\circ + f(x^\circ)) = r[\nabla_H f(x^\circ) + Q](0, y^\circ).$$

Corollary 4.7: *Suppose that in the conditions of Corollary 4.6, Y is a Banach space, Q is convex, and $-f(x^\circ) \in Q$. Then $f + Q$ is metrically regular along H near $(x^\circ, 0)$ if and only if*

$$0 \in \operatorname{int}\{f(x^\circ) + \nabla_H f(x^\circ) X + Q\}.$$

5. Metric regularity for mappings into product spaces

Let $F : X \rightrightarrows Y$ be a mapping between a normed linear space X and the Cartesian product of $n \ge 1$ normed linear spaces $Y = Y_1 \times Y_2 \times \ldots \times Y_n$. Suppose that F can be represented as $F = (F_1, F_2, \ldots, F_n)$, where each F_i is a mapping from X into Y_i. This means that for any $x \in X$ its image $F(x)$ under F is the product of the images: $F(x) = F_1(x) \times F_2(x) \times \ldots \times F_n(x)$. If F is single-valued this assumption is fulfilled automatically.

Let $y^\circ = (y_1^\circ, y_2^\circ, \dots, y_n^\circ) \in F(x^\circ)$.

Besides considering metric regularity and pseudo-Lipschitz properties of F, one can also examine these properties componentwise.

As it follows from the next proposition, the pseudo-Lipschitz property of F is equivalent to all its components enjoying the same property.

Proposition 5.1:

(i) $l_\delta[F](x^\circ, y^\circ) = \max\limits_{1\le i\le n} l_\delta[F_i](x^\circ, y_i^\circ)$ *for any* $\delta > 0$*;*

(ii) $l[F](x^\circ, y^\circ) = \max\limits_{1\le i\le n} l[F_i](x^\circ, y_i^\circ)$.

Proof: The first assertion follows from (12) and the definition of the norm in the product space:

$$l_\delta[F](x^\circ, y^\circ) = \sup_{\substack{x,x'\in B_\delta(x^\circ)\\ y_i\in F_i(x')\cap B_\delta(y_i^\circ)\\ i=1,2,\dots,n}} \left[\frac{\max\limits_{1\le i\le n} d(y_i, F_i(x))}{d(x,x')}\right]_0$$

$$= \max_{1\le i\le n} \sup_{\substack{x,x'\in B_\delta(x^\circ)\\ y_i\in F_i(x')\cap B_\delta(y_i^\circ)}} \left[\frac{d(y_i, F_i(x))}{d(x,x')}\right]_0 = \max_{1\le i\le n} l_\delta[F_i](x^\circ, y_i^\circ).$$

The second assertion is a corollary of the first one. □

Under some mild assumptions metric regularity of F implies metric regularity of all it components.

Proposition 5.2:

(i) If $F(x) \cap B_\delta(y^\circ) \ne \emptyset$ *for some* $\delta > 0$ *and all* $x \in B_\delta(x^\circ)$*, then*

$$r_\delta[F](x^\circ, y^\circ) \le \min_{1\le i\le n} r_\delta[F_i](x^\circ, y_i^\circ);$$

(ii) If $F(x) \cap B_\delta(y^\circ) \ne \emptyset$ *for all sufficiently small* $\delta > 0$ *and all* $x \in B_\delta(x^\circ)$*, then*

$$r[F](x^\circ, y^\circ) \le \min_{1\le i\le n} r[F_i](x^\circ, y_i^\circ).$$

Proof: Let the assumptions of (i) be fulfilled with a certain $\delta > 0$. Take any i and any $x \in B_\delta(x^\circ)$ and $y_i \in B_\delta(y_i^\circ)$. For all $j \ne i$ take some $y_j \in F_j(x) \cap B_\delta(y_j^\circ)$ and compose $y = (y_1, y_2, \dots, y_n)$. Then $y \in B_\delta(y^\circ)$, $d(y, F(x)) = d(y_i, F_i(x))$ and $d(x, F^{-1}(y)) = d(x, F_i^{-1}(y_i))$. By the definition (11),

$$r_\delta[F](x^\circ, y^\circ) \le \left[\frac{d(y, F(x))}{d(x, F^{-1}(y))}\right]_\infty = \left[\frac{d(y_i, F_i(x))}{d(x, F_i^{-1}(y_i))}\right]_\infty.$$

Taking infimum over $x \in B_\delta(x^\circ)$ and $y_i \in B_\delta(y_i^\circ)$ in the right hand side of the last relation, we have $r_\delta[F](x^\circ, y^\circ) \le r_\delta[F_i](x^\circ, y_i^\circ)$. Since this inequality is valid for any i, assertion (i) holds true. The second assertion is a corollary of the first one. □

The reverse of Proposition 5.2 does not hold.

Example 5.3 Consider $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ with $f_1(x_1, x_2) = x_1 + x_2$, $f_2(x_1, x_2) = 2x_1 + 2x_2$. Evidently, $r[f_1](0,0) = 1$, $r[f_2](0,0) = 2$ while $r[f](0,0) = 0$.

There is another way of dealing with mappings into product spaces. The following local regularity property of F near $(x^\circ, y^\circ) \in \operatorname{gph} F$, taking into account the behaviour of its components, can be of interest:

- *metric multi-regularity*: there exist $\kappa > 0$ and $\delta > 0$ such that

$$d(0, \bigcap_{i=1}^{n} (F_i^{-1}(y_i) - x_i)) \le \kappa \max_{1 \le i \le n} d(y_i, F_i(x_i)),$$
$$\forall x_i \in B_\delta(x^\circ),\ y_i \in B_\delta(y_i^\circ),\ i = 1, 2, \ldots, n.$$

Obviously, when $n = 1$ the above property coincides with the traditional one described in Section 2. When $n > 1$ the components F_i describe different constraints (groups of constraints). The above condition allows for independent perturbations of these constraints (both the right-hand sides and variables) and guarantees the existence of a joint "stabilizing" action and provides an "error bound" type estimate.

An interpretation of the metric multi-regularity property can be made in terms of the following consistence problem:

$$x \in f_1^{-1}(\Theta_1), \quad x \in f_2^{-1}(\Theta_2),$$

where Θ_1 and Θ_2 correspond to two different loads applied independently to a mechanical structure and are subject to perturbations. The reactions of the structure to these perturbations are also independent. A stable design is needed with respect to both loads. Defining $F_1(x) = f_1(x) - \Theta_1$, $F_2(x) = f_2(x) - \Theta_2$, one can interpret the desired stability property in terms of metric multi-regularity.

Unlike the preceding sections, here we formulate only the regularity property. As one can see from Proposition 5.4 below, its Lipschitz-like counterpart coincides with the conventional pseudo-Lipschitzness considered in Section 2.

The following constants are convenient for characterizing the above regularity property:

$$\hat{r}_\delta[F](x^\circ, y^\circ) = \inf_{\substack{x_i \in B_\delta(x^\circ),\, y_i \in B_\delta(y_i^\circ) \\ i=1,2,\ldots,n}} \left[\frac{\max\limits_{1 \le i \le n} d(y_i, F_i(x_i))}{d(0, \bigcap\limits_{i=1}^{n} (F_i^{-1}(y_i) - x_i))} \right]_\infty, \tag{34}$$

$$\hat{r}[F](x^\circ, y^\circ) = \liminf_{\substack{x_i \to x^\circ,\, y_i \to y_i^\circ \\ i=1,2,\ldots,n}} \left[\frac{\max\limits_{1 \le i \le n} d(y_i, F_i(x_i))}{d(0, \bigcap\limits_{i=1}^{n} (F_i^{-1}(y_i) - x_i)} \right]_\infty. \tag{35}$$

The constants (11), (17) correspond to taking $x_1 = x_2 = \ldots = x_n$ in the right hand sides of the above formulas. Thus, in general

$$\hat{r}_\delta[F](x^\circ, y^\circ) \le r_\delta[F](x^\circ, y^\circ), \qquad \hat{r}[F](x^\circ, y^\circ) \le r[F](x^\circ, y^\circ).$$

Consequently, considered in this section metric multi-regularity is in general stronger than the corresponding conventional property from Section 2.

The "Lipschitz-like" counterparts of (34) and (35) can be written as follows:

$$\hat{l}_\delta[F](x^\circ, y^\circ) = \sup_{\substack{h \in X,\ x_i, x_i+h \in B_\delta(x^\circ) \\ y_i \in F_i(x_i+h) \cap B_\delta(y_i^\circ) \\ i=1,2,\ldots,n}} \left[\frac{\max\limits_{1 \le i \le n} d(y_i, F_i(x_i))}{\|h\|} \right]_0, \tag{36}$$

$$\hat{l}[F](x^\circ, y^\circ) = \limsup_{\substack{h \to 0,\ x_i \to x^\circ,\ y_i \to y_i^\circ \\ y_i \in F_i(x_i+h),\ i=1,2,\ldots,n}} \left[\frac{\max\limits_{1 \le i \le n} d(y_i, F_i(x_i))}{\|h\|} \right]_0.$$

They coincide with the corresponding constants from Section 2.

Proposition 5.4:

(i) $\hat{l}_\delta[F](x^\circ, y^\circ) = l_\delta[F](x^\circ, y^\circ)$ *for any* $\delta > 0$*;*
(ii) $\hat{l}[F](x^\circ, y^\circ) = l[F](x^\circ, y^\circ)$*.*

Proof: The inequality

$$\hat{l}_\delta[F](x^\circ, y^\circ) \ge l_\delta[F](x^\circ, y^\circ)$$

is straightforward. On the other hand, by the definition (12),

$$d(y, F(x)) \le l_\delta[F](x^\circ, y^\circ)\|h\|$$

for any $x \in B_\delta(x^\circ)$, $h \in B_\delta$ such that $x+h \in B_\delta(x^\circ)$, and any $y \in F(x+h) \cap B_\delta(y^\circ)$. Then by the definition of distance in product spaces, for any i,

$$d(y_i, F_i(x_i)) \le l_\delta[F](x^\circ, y^\circ)\|h\|$$

for any $x_i \in B_\delta(x^\circ)$, $h \in B_\delta$ such that $x_i + h \in B_\delta(x^\circ)$, and any $y_i \in F_i(x_i + h) \cap B_\delta(y_i^\circ)$, and consequently,

$$\max_{1 \le i \le n} d(y_i, F_i(x_i)) \le l_\delta[F](x^\circ, y^\circ)\|h\|.$$

By the definition (36), the last inequality implies $\hat{l}_\delta[F](x^\circ, y^\circ) \le l_\delta[F](x^\circ, y^\circ)$. This proves the first assertion. The second assertion is a corollary of the first one. □

As usual we write $\hat{r}_\delta[f](x^\circ)$ and $\hat{r}[f](x^\circ)$ if $f : X \to Y$ is single-valued.

The properties considered in this section can be treated in the framework of the "along a subspace" properties examined in Section 4. Namely, let $Z = X^n$ and $z = (x_1, x_2, \ldots, x_n) \in Z$. One can consider the mapping $\Phi :\ Z \rightrightarrows Y$ defined by

$$\Phi(z) = F_1(x_1) \times F_2(x_2) \times \ldots \times F_n(x_n). \tag{37}$$

Note that each "component" of Φ in the above formula depends on its own argument, and $(z^\circ, y^\circ) \in \operatorname{gph} \Phi$, where $z^\circ = (x^\circ, x^\circ, \ldots, x^\circ)$.

In the space Z, consider the diagonal subspace

$$H = \{(x_1, x_2, \ldots, x_n) \in X^n :\ x_1 = x_2 = \ldots = x_n\}. \tag{38}$$

Evidently, $\Phi(z) = F(x)$ if $z = (x, x, \ldots, x) \in H$.

The next Proposition shows that metric regularity (pseudo-Lipschitz property) of Φ near (z°, y°) along H is exactly metric multi-regularity (pseudo-Lipschitz property) of F near (x°, y°). Another interesting feature of Φ, which follows from part (v) of the next Proposition, is that pseudo-Lipschitz property of Φ is equivalent to its pseudo-Lipschitz property along H.

Proposition 5.5: *Let the mapping $\Phi : Z \rightrightarrows Y$ be defined by* (37) *and let* $\delta > 0$. *Then*

(i) $\hat{r}_\delta[F](x^\circ, y^\circ) = r_{H,\delta}[\Phi](z^\circ, y^\circ)$;
(ii) $l_{\delta/2}[F](x^\circ, y^\circ) \le l_{H,\delta}[\Phi](z^\circ, y^\circ) \le l_{2\delta}[F](x^\circ, y^\circ)$;
(iii) $l_\delta[F](x^\circ, y^\circ) = l_\delta[\Phi](z^\circ, y^\circ)$;
(iv) $\hat{r}[F](x^\circ, y^\circ) = r_H[\Phi](x^\circ, y^\circ)$;
(v) $l[F](x^\circ, y^\circ) = l_H[\Phi](x^\circ, y^\circ) = l[\Phi](x^\circ, y^\circ)$.

Proof: It follows immediately from the definition (37) that, for any $z = (x_1, x_2, \dots, x_n) \in Z$, $y = (y_1, y_2, \dots, y_n) \in Y$, one has

$$d(y, \Phi(z)) = \max_{1\le i\le n} d(y_i, F_i(x_i)),$$
$$\Phi^{-1}(y) = \{(x_1, x_2, \dots, x_n) \in Z : x_i \in F_i^{-1}(y_i),\ i = 1, 2, \dots, n\},$$
$$d_H(z, \Phi^{-1}(y)) = d(0, \bigcap_{i=1}^{n} (F_i^{-1}(y_i) - x_i)).$$

Comparing the definitions (28) and (34) proves the first assertion.

The second assertion follows from comparing (29) and (36) due to the elementary estimates: $\|h\| \le \delta$ if $x_i, x_i + h \in B_{\delta/2}(x^\circ)$ and $x_i + h \in B_{2\delta}(x^\circ)$ if $x_i \in B_\delta(x^\circ)$ and $\|h\| \le \delta$.

To prove (iii) we first use the definitions (12) and (37) and denotations $z = (x_1, x_2, \dots, x_n)$, $z' = (x'_1, x'_2, \dots, x'_n)$, $y = (y_1, y_2, \dots, y_n)$:

$$l_\delta[\Phi](z^\circ, y^\circ) = \sup_{\substack{z, z' \in B_\delta(z^\circ) \\ y \in F(z') \cap B_\delta(y^\circ)}} \left[\frac{\max\limits_{1\le i\le n} d(y_i, F_i(x_i))}{d(z, z')} \right]_0$$
$$= \max_{1\le i\le n} \sup_{\substack{z, z' \in B_\delta(z^\circ) \\ y_i \in F_i(x'_i) \cap B_\delta(y^\circ)}} \left[\frac{d(y_i, F_i(x_i))}{d(z, z')} \right]_0,$$

The maximum in the right hand side of the above formula is achieved at some index i. For this index, when considering z and z' in the supremum, it is natural to take $x_j = x'_j$ if $j \ne i$. Thus, this supremum equals $l_\delta[F_i](x^\circ, y_i^\circ)$, and the assertion follows from Proposition 5.1.

The equalities (iv) and (v) are corollaries of (i)–(iii) and Proposition 5.4. □

Another important property inherited by Φ from F is strict differentiability. Consider a single-valued mapping $f = (f_1, f_2, \dots, f_n)$ from X into $Y = Y_1 \times Y_2 \times \dots \times Y_n$ and define another mapping ϕ from $Z = X^n$ into Y by

$$\phi(z) = (f_1(x_1), f_2(x_2), \dots, f_n(x_n)), \tag{39}$$

where $z = (x_1, x_2, \dots, x_n) \in Z$.

Proposition 5.6: *The following conditions are equivalent:*

(i) f is strictly differentiable at x° with the derivative

$$\nabla f(x^\circ) = (\nabla f_1(x^\circ), \nabla f_2(x^\circ), \ldots, \nabla f_n(x^\circ));$$

(ii) ϕ is strictly differentiable at z° with the derivative $\nabla\phi(z^\circ)$ given by

$$\nabla\phi(z^\circ)z = (\nabla f_1(x^\circ)x_1, \nabla f_2(x^\circ)x_2, \ldots, \nabla f_n(x^\circ)x_n). \tag{40}$$

Proof: Let (i) holds. Consider a linear and continuous mapping $\nabla\phi(x^\circ)$ on Z defined by (40). Then for any $z = (x_1, x_2, \ldots, x_n)$, $z' = (x_1', x_2', \ldots, x_n') \in Z$, $z \neq z'$ one has

$$\begin{aligned} &\frac{\|\phi(z') - \phi(z) - \nabla\phi(z^\circ)(z'-z)\|}{\|z'-z\|} \\ &= \max_{1\le i\le n} \frac{\|f_i(x_i') - f_i(x_i) - \nabla f_i(x^\circ)(x_i'-x_i)\|}{\|z'-z\|} \\ &\le \max_{1\le i\le n} \left[\frac{\|f_i(x_i') - f_i(x_i) - \nabla f_i(x^\circ)(x_i'-x_i)\|}{\|x_i'-x_i\|}\right]_0 . \end{aligned}$$

Consequently, ϕ is strictly differentiable at z° with the derivative $\nabla\phi(x^\circ)$.

Let ϕ be strictly differentiable at z°. Then, each its component ϕ_i is also strictly differentiable at z°, that is there exists a linear and continuous mapping $l_i : Z \to Y_i$ such that

$$\lim_{\substack{z,z'\to z^\circ \\ z'\neq z}} \frac{\phi_i(z') - \phi_i(z) - l_i(z'-z)}{\|z'-z\|} = 0.$$

Taking into account that l_i must have a representation $l_i(z) = l_{i1}(x_1) + l_{i2}(x_2) + \ldots + l_{in}(x_n)$, where for each $j = 1, 2, \ldots, n$, l_{ij} is a linear and continuous mapping from X into Y_i and $z = (x_1, x_2, \ldots, x_n)$, and $\phi_i(z) = f_i(x_i)$, the last equality implies that $l_{ij} = 0$ if $j \neq i$ and f_i is strictly differentiable at x° with l_{ii} being its strict derivative. Thus, $l_i(z) = \nabla f_i(x^\circ)x_i$. Consequently, the strict derivative of ϕ can be represented as (40). □

Application of Theorem 4.2 to the setting adopted in this section yields the following statement.

Theorem 5.7: *Let $F : X \rightrightarrows Y$ and $g : X \to Y$, where X is a Banach space and $Y = Y_1 \times Y_2 \times \ldots \times Y_n$ is the Cartesian product of $n \ge 1$ normed linear spaces. Suppose that $F = (F_1, F_2, \ldots, F_n)$, where each F_i is a mapping from X into Y_i with closed graph. Let $(x^\circ, y^\circ) \in \operatorname{gph} F$. Then*

$$|\hat{r}_\delta[F+g](x^\circ, y^\circ + g(x^\circ)) - \hat{r}_\delta[F](x^\circ, y^\circ)| \le l_{2\delta}[g](x^\circ).$$

Corollary 5.8: *Under the conditions of Theorem 5.7*

$$|\hat{r}[F+g](x^\circ, y^\circ + g(x^\circ)) - \hat{r}[F](x^\circ, y^\circ)| \le l[g](x^\circ).$$

Corollary 5.9: *Let $l[g](x^\circ) = 0$ in the conditions of Corollary 5.8. Then*

$$\hat{r}[F+g](x^\circ, y^\circ + g(x^\circ)) = \hat{r}[F](x^\circ, y^\circ).$$

The next "regularity statement" follows from Theorem 4.5.

Theorem 5.10: *Let $f : X \to Y$ and $F : X \rightrightarrows Y$, where X is a Banach space and Y is the Cartesian product of $n \geq 1$ normed spaces. Suppose that $\mathrm{gph}\, F$ is closed in $X \times Y$, $(0, y^\circ) \in \mathrm{gph}\, F$ and f is strictly differentiable at x°. Let $Z = X^n$, $z^\circ = (x^\circ, x^\circ, \ldots, x^\circ) \in Z$ and let the subspace $H \subset Z$ be defined by (38). Define the mappings $\hat{F} : H \rightrightarrows Y$ as $\hat{F}(x, x, \ldots, x) = F(x)$, $x \in X$, and $\hat{\phi} : Z \times H \to Y$ as $\hat{\phi}(z, h) = \phi(z + h)$, $h \in H$, $z \in Z$, where ϕ is defined by (39). Then*

$$r[z^\circ; \hat{\phi} + \hat{F}](0, y^\circ + f(x^\circ)) = r[\nabla f(x^\circ) + F](0, y^\circ).$$

Proof: The only thing which needs to be checked in this theorem is the equality $r[\nabla_H \phi(z^\circ) + \hat{F}](0_H, y^\circ) = r[\nabla f(x^\circ) + F](0, y^\circ)$. It follows from the definitions and Proposition 5.6. □

Corollary 5.11: *Let $f : X \to Y$ be a function from a Banach space into the Cartesian product of $n \geq 1$ normed spaces $Y = Y_1 \times Y_2 \times \ldots \times Y_n$, and Q be a closed subset of Y. Suppose that f is strictly differentiable at x°, $y^\circ \in Q$ and $Q = Q_1 \times Q_2 \times \ldots \times Q_n$ with $Q_i \subset Y_i$, $i = 1, 2, \ldots, n$. Then*

$$\hat{r}[f + Q](x^\circ, y^\circ + f(x^\circ)) = r[\nabla f(x^\circ) + Q](0, y^\circ).$$

Corollary 5.12: *Suppose that in the conditions of Corollary 5.11, Y is a Banach space, Q is convex, and $-f(x^\circ) \in Q$. Then $f + Q$ is metrically multi-regular near $(x^\circ, 0)$ if and only if*

$$0 \in \mathrm{int}\, \{f(x^\circ) + \nabla f(x^\circ) X + Q\}. \tag{41}$$

6. Systems of generalized equations

The regularity results of the last section for mappings into product spaces are applicable to solvability of systems of generalized equations.

If $f = (f_1, f_2, \ldots, f_n)$ and $Q = Q_1 \times Q_2 \times \ldots \times Q_n$, where $f_i : X \to Y_i$, $Q_i \subset Y_i$, $i = 1, 2, \ldots, n$, then the inclusion $-f(x^\circ) \in Q$ in Corollary 5.12 means that x° is a solution of the system

$$f_i(u) + Q_i \ni 0, \quad i = 1, 2, \ldots, n. \tag{42}$$

In the case $Q_i = \{0\}$ the corresponding inclusion reduces to a usual equation. If Q_i is a cone (of positive elements) in Y_i then the corresponding inclusion can be treated as an inequality.

Let x° be a solution of system (42). We say that (42) is *metrically multi-regular* at x° if there exist $\kappa > 0$ and $\delta > 0$ such that for any $x_i \in B_\delta(x^\circ)$ and $y_i \in B_\delta$, $i = 1, 2, \ldots, n$, there exists $x \in X$ such that

$$f_i(x_i + x) + Q_i \ni y_i, \quad i = 1, 2, \ldots, n, \tag{43}$$

$$\|x\| \leq \kappa \max_{1 \leq i \leq n} d(y_i - f_i(x_i), Q_i). \tag{44}$$

The metric multi-regularity property guarantees the existence of a solution of the perturbed system (43) and provides the estimate (44) of the norm of the solution which can be interpreted as an *error bound*. Note that both the right-hand sides and variables are perturbed in (43) and the variables in different generalized equations are perturbed independently.

One can easily check that metric multi-regularity of (42) at x° as it is defined above is nothing else but metric multi-regularity near (x°, y°) (with $y^\circ = 0$) of the mapping $F : X \rightrightarrows Y$ defined by $F(x) = f(x) + Q$, where $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ and $Q = Q_1 \times Q_2 \times \ldots \times Q_n$.

The realization of Corollary 5.12 produces the following theorem from [5]:

Theorem 6.1: *Let $x^\circ \in X$ be a solution of system* (42), *where the functions $f_i : X \to Y_i$ between Banach spaces are strictly differentiable at x° and the sets $Q_i \subset Y_i$ are closed and convex. Then system* (42) *is metrically multi-regular at x° if and only if* (41) *holds true.*

The next proposition from [5] gives an estimate for the solution of the system of perturbed generalized equations, in which only the variables are perturbed, not the right-hand sides of the equations. Here, the variable in each equation is perturbed not relative to the solution of the whole system, but relative to any sufficiently close to it solution of this particular equation.

Proposition 6.2: *Let $x^\circ \in X$ be a solution of system* (42). *Suppose that, in the neighborhood of x°, the functions f_i are Lipschitz continuous for $i \in I(x^\circ) = \{i \in \{1, 2, \ldots, n\} : -f_i(x^\circ) \in \operatorname{bd} Q_i\}$ and continuous for $i \notin I(x^\circ)$. If system* (42) *is metrically multi-regular at x° then there exist constants $L > 0$ and $\delta > 0$ such that for any $x_i \in B_\delta(x^\circ)$ satisfying the individual inclusions*

$$f_i(x_i) + Q_i \ni 0, \quad i = 1, 2, \ldots, n, \tag{45}$$

and any variations $u_i \in B_\delta$, $i = 1, 2, \ldots, n$, one can find $x \in X$ such that

$$f_i(x_i + u_i + x) + Q_i \ni 0, \quad i = 1, 2, \ldots, n,$$
$$\|x\| \le L \max_{i \in I(x^\circ)} \|u_i\|.$$

It follows from Theorem 6.1 that in the case when all the spaces are Banach and the functions are strictly differentiable, the regularity condition (41) is sufficient for the conclusions of Proposition 6.2.

There exists a close relationship between metric multi-regularity of system (42) and strong regularity of collections of sets considered in [16, 17] in the case when the sets are given in the form (45); see [5] for details.

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