" DYNAMICAL SYSTEMS:
STABILITY, CONTROL, OPTIMIZATION "

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ON NECESSARY AND SUFFICIENT EXTREMALITY CONDITIONS

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1. The generalized Euler equation. Let X be a Banach space and let $\Omega_1, \Omega_2, \dots, \Omega_n$ ($n \geq 2$) be nonempty closed sets in X. The following definition was introduced in [12] (The current formulation follows [4]).

The system of sets Ω_i, $i = 1, 2, \dots, n$, is called *extremal*, if $\cap_{i=1}^n \Omega_i \neq \emptyset$ and there exist such sequences $\{a_k^i\} \subset X$, $i = 1, 2, \dots, n$, converging to zero, that $\bigcap_{i=1}^n \Omega_{ik} = \emptyset$, where $\Omega_{ik} = \Omega_i - a_k^i = \{x - a_k^i : x \in \Omega_i\}$, $i = 1, 2, \dots, n$, $k = 1, 2, \dots$. Any point $x \in \bigcap_{i=1}^n \Omega_i$ is called an *extremal point* of the system of sets Ω_i, $i = 1, 2, \dots, n$.

The concept of extremal system characterizes mutual arrangement of sets in space. In case of an Asplund space it is possible to formulate a dual criterion of extremality (see [4, 12, 13, 15, 16]).

Theorem 1. *Let X be an Asplund space and let x be an extremal point of the system of sets Ω_i, $i = 1, 2, \dots, n$. Then for any $\varepsilon > 0$, $\delta > 0$ there are such elements $x_i^* \in \hat{N}_\delta(x|\Omega_i)$, $i = 1, 2, \dots, n$, that*

$$\|x_1^* + x_2^* + \dots + x_n^*\|_* < \varepsilon, \tag{1}$$

$$\|x_1^*\|_* + \|x_2^*\|_* + \dots + \|x_n^*\|_* = 1. \tag{2}$$

In theorem 1 the *strict δ-normal cones* are used. They are defined as follows

$$\hat{N}_\delta(x|\Omega) = \bigcup [N(u|cl\Omega) : \ u \in cl\Omega \cap B_\delta(x)],$$

$$N(x|\Omega) = \left\{ x^* \in X^* : \ \limsup_{\substack{u \to x \\ u \in \Omega}} \frac{< x^*, u - x >}{|u - x|} \leq 0 \right\}.$$

If $n = 2$ theorem 1 can be viewed as a certain generalization of the classical separation theorem for convex sets. The condition (1) is called the *generalized Euler equation*.

Asplund spaces form a rather broad class of Banach spaces. It includes, in particular, all Banach spaces with Fréchet differentiable renorms and all spaces with separable duals. Theorem 1 gives extremal characterization of Asplund spaces. In fact this characterization is necessary and sufficient for a space to be Asplund (see [15]).

The generalized Euler equation was used in [4 – 7, 12, 13, 16] as the basic tool for deducing the calculus formulas and necessary conditions of an extremum, replacing thus the traditional separation theorem.

The next result generalizing the Bishop-Phelps density theorem follows immediately from theorem 1.

Corollary 1.1. *Let X be an Asplund space, let Ω be a closed set in X, and let x be a boundary point of Ω. Then $\hat{N}_\delta(x|\Omega) \neq \{0\}$ for any $\delta > 0$.*

2. Extremal sets systems (case $n = 2$). The generalized Euler equation gives necessary extremality conditions for systems of sets. It is also true for some systems not being extremal in the sense of the above definition. It is of interest to find out for which possibly broader class of sets the generalized Euler equation is valid.

First we shall consider the case $n = 2$. Following [9] let us introduce for sets Ω_1 and Ω_2 a constant

$$\theta(\Omega_1, \Omega_2) = \sup\{r > 0 : \ B_r \subset \Omega_1 - \Omega_2\}, \tag{3}$$

describing the measure of their "overlapping". The difference of sets in (3) is understood in the algebraic sense: $\Omega_1 - \Omega_2 = \{\omega_1 - \omega_2 : \ \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}$. In case $\{r > 0 : \ B_r \subset \Omega_1 - \Omega_2\} = \emptyset$ we shall write $\theta(\Omega_1, \Omega_2) = 0$.

Let us say that the sets Ω_1, Ω_2 form an *extremal system in a neighborhood* of a point $x \in \Omega_1 \cap \Omega_2$ if

$$\liminf_{\substack{\omega_1 \in \Omega_1,\ \omega_2 \in \Omega_2 \\ \omega_1 \to x,\ \omega_2 \to x \\ \rho \to +0}} \frac{\theta([\Omega_1 - \omega_1] \cap B_\rho,\ [\Omega_2 - \omega_2] \cap B_\rho)}{\rho} = 0. \tag{4}$$

Let us introduce two local constants based on (3):

$$\dot{\theta}_x(\Omega_1, \Omega_2) = \liminf_{\rho \to +0} \frac{\theta(\Omega_1 \cap B_\rho(x),\ \Omega_2 \cap B_\rho(x))}{\rho}, \tag{5}$$

$$\hat{\theta}_x(\Omega_1, \Omega_2) = \liminf_{\substack{\omega_1 \in \Omega_1,\ \omega_2 \in \Omega_2 \\ \omega_1 \to x,\ \omega_2 \to x}} \theta_0(\Omega_1 - \omega_1,\ \Omega_2 - \omega_2). \tag{6}$$

Then condition (4) is equivalent to the equality $\hat{\theta}_x(\Omega_1, \Omega_2) = 0$.

If the system of sets Ω_1, Ω_2 is extremal in the sense of the definition of section 1 (which is obviously equivalent to the condition $\theta(\Omega_1, \Omega_2) = 0$), the sets Ω_1, Ω_2 form extremal system in a neighborhood of a point x.

In general, the definition (4) differs from the previous in two important moments. First, it is required not strict "nonoverlapping" of sets but up to an indefinitely small deformation (see (5)). Second, the sets can "nonoverlap" not in the point x: it is sufficient that in any its neighborhood there exist such points $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$, that the sets $\Omega_1 - \omega_1$, $\Omega_2 - \omega_2$ "almost nonoverlap" (see (6)).

In the convex case the definition of extremality of sets takes a natural form.

P r o p o s i t i o n 1. *Let Ω_1, Ω_2 be convex sets, $\mathrm{int}\Omega_2 \neq \emptyset$. The sets Ω_1, Ω_2 form an extremal system in a neighborhood of a point $x \in \Omega_1 \cap \Omega_2$ if and only if $\Omega_1 \cap \mathrm{int}\Omega_2 = \emptyset$.*

The definition (4) is a restatement of the requirement, that a function $(\omega_1, \omega_2) \to \omega_2 - \omega_1 :$ $\Omega_1 \times \Omega_2 \to X$ not covers [8] in a neighborhood of a point x. Application of a slight modification of the covering theorem from [8] gives the following result, representing some generalization of the separation theorem.

T h e o r e m 2. *Let X be an Asplund space. The sets Ω_1, Ω_2 form an extremal system in a neighborhood of a point x if and only if for any $\delta > 0$ there exists a nonnull element $x^* \in X^*$, such that $x^* \in \hat{N}_\delta(x|\Omega_1)$, $\ -x^* \in \hat{N}_\delta(x|\Omega_2)$.*

The above theorem can serve as a tool for deducing necessary and sufficient conditions of "almost extremality" in various extremal problems.

3. Extremal sets systems (case $n \geq 2$). Let us say that the sets Ω_i, $i = 1, 2, \ldots, n$, form an *extremal system in a neighborhood* of a point x if the sets $\tilde{\Omega}_1 = \prod_{i=1}^n \Omega_i$ and $\tilde{\Omega}_2 = \{(\omega, \omega, \ldots, \omega) \in X^n\}$ form an extremal system in a neighborhood of a point $(x, x, \ldots, x) \in X^n$.

Let us denote $I = \{1, 2, \ldots, n\}$. The following proposition gives an equivalent definition of the extremal sets system.

P r o p o s i t i o n 2. *The sets Ω_i, $i \in I$, form an extremal system in a neighborhood of a point x if and only if for any $j \in I$ the sets $\tilde{\Omega}_1 = \prod\limits_{i \in I \setminus j} \Omega_i$ and $\tilde{\Omega}_2 = \{(\omega, \omega, \ldots, \omega) \in X^{n-1} : \ \omega \in \Omega_j\}$ form an extremal system in a neighborhood of a point $(x, x, \ldots, x) \in X^{n-1}$.*

The following theorem gives dual criterion of extremality (the generalized Euler equation).

Theorem 3. *Let X be an Asplund space. The sets Ω_i, $i = 1,2,\ldots,n$, form an extremal system in a neighborhood of a point x if and only if for any $\delta > 0$ there exist such elements $x_i^* \in \hat{N}_\delta(x|\Omega_i)$, $i = 1,2,\ldots,n$, that conditions (1), (2) are true.*

4. The covering theorem. When proving dual extremality criteria the central role is played by the covering theorem [8 – 10], which can be viewed as a some generalization of the classical open mapping theorem to the case of a set-valued mapping. It also generalizes and strengthens the corresponding result in [2]. For investigating extremality problems the following equivalent formulation of the covering property is convenient.

Let us consider an upper semicontinuous set-valued mapping $F:\ \Omega \to 2^Y$ (its graph $G(F)$ is closed in $X \times Y$). Let us denote

$$\delta_F(u,v,\rho) = \sup\{r > 0 : B_r(v) \subset F(\Omega \cap B_\rho(u))\}. \tag{7}$$

In case $\{r > 0 : B_r(v) \subset F(\Omega \cap B_\rho(u))\} = \emptyset$ we shall write $\delta_F(u,v,\rho) = 0$.

Let us say, that a set-valued mapping F covers (at a linear rate [14]) in a neighborhood of $(x,y) \in G(F)$ if

$$\liminf_{\substack{u\in\Omega,\ v\in F(u)\\ u\to x,\ v\to y\\ \rho\to +0}} \frac{\delta_F(u,v,\rho)}{\rho} > 0. \tag{8}$$

For characterizing the covering property the following constant is used

$$b_F(x,y) = \sup_{\delta>0} \inf\{\|x^*\|_* :\ x^* \in \hat{\partial}_\delta F(x,y;y^*),\ \|y^*\|_* = 1\}, \tag{9}$$

where $\hat{\partial}_\delta F(x,y;\cdot):\ Y^* \to X^*$ is a *strict δ-coderivative* of F at (x,y). It is defined by the equality

$$\hat{\partial}_\delta F(x,y;y^*) = \{x^* \in X^* :\ (x^*,-y^*) \in \hat{N}_\delta(x,\bar{y}|G(F))\}.$$

Theorem 4. *Let Y be an Asplund space. Then F covers in a neighborhood of $(x,y) \in G(F)$ if and only if $b_F(x,y) > 0$.*

In case of an ordinary (single-valued) continuous mapping $f:\ \Omega \to Y$ the following constant is used in the definition of the covering property instead of (7):

$$\delta_f(u,\rho) = sup\{r > 0 : B_r(f(u)) \subset f(\Omega \cap B_\rho(u))\},$$

and in the formulation of the covering criterion instead of (9) it is necessary to use a constant

$$b_f(x) = \sup_{\delta>0} \inf\{\|x^*\|_* :\ x^* \in \hat{\partial}_\delta^- < y^*, f > (x),\ \|y^*\|_* = 1\}. \tag{10}$$

In (10) the following notations are used:

$$< y^*, f > (u) = < y^*, f(u) >,\ u \in \Omega,$$

$$\hat{\partial}_\delta^- \varphi(x) = \bigcup \left[\partial^-(cl^1\varphi)(u) :\ u \in cl\Omega \cap B_\delta(x),\ |cl^1\varphi(u) - \varphi(x)| \le \delta\right], \tag{11}$$

$$\partial^-\varphi(x) = \left\{x^* \in X^* :\ \liminf_{\substack{u\to x\\ u\in\Omega}} \frac{\varphi(u) - \varphi(x) - < x^*, u - x >}{\|u - x\|} \ge 0\right\}. \tag{12}$$

The sets (11), (12) are called accordingly a *strict lower δ-semidifferential* and a *lower semidifferential* of $\varphi:\ \Omega \to R$ at x.

Theorem 5. *Let Y be an Asplund space. Then f covers in a neighborhood of $x \in \Omega$ if and only if $b_f(x) > 0$.*

References

1. *Borwein J.M., Preiss D.* A smooth variational principle with application to subdifferentiability and differentiability of convex functions // Trans. Amer. Math. Soc. 1987. Vol. 303. P. 517 – 527.

2. *Dmitruk A. V., Miljutin A. A., Osmolovskii N. P.* Ljusternik's theorem and the theory of extrema // Russ. Math. Surv. 1980. Vol. 35. P. 11 – 51.

3. *Fabian M.* Subdifferentiability and trustworthiness in the light of a new variational principle of Borwein and Preiss // Acta Univ. Carolinae. 1989. Vol. 30. P. 51 – 56.

4. *Kruger A. Ya.* Generalized differentials of nonsmooth functions. Minsk, 1981. Deposited in VINITI No. 1332-81 (in Russian).

5. *Kruger A. Ya.* Necessary conditions of an extremum in problems of nonsmooth optimization. Minsk, 1981. Deposited in VINITI No. 1333-81 (in Russian).

6. *Kruger A. Ya.* Generalized differentials of nonsmooth functions and necessary conditions for an extremum // Siberian Math. J. 1985. Vol. 26. P. 370 – 379.

7. *Kruger A. Ya.* Properties of generalized differentials // Siberian Math. J. 1985. Vol. 26. P. 822 – 832.

8. *Kruger A. Ya.* Covering theorem for set-valued mappings // Optimization. 1988. Vol. 19. P. 763 – 780.

9. *Kruger A. Ya.* Strict ε-semidifferentials and differentiating set-valued mappings // Dokl. Akad. Nauk Belarus. 1996. Vol. 40, N. 6. P. 38 – 43 (in Russian).

10. *Kruger A. Ya.* Strict ε-semidifferentials and extremality conditions // Dokl. Akad. Nauk Belarus. 1997. Vol. 41, N. 3. P. 21 – 26 (in Russian).

11. *Kruger A. Ya.* On extremality of sets systems // Dokl. Akad. Nauk Belarus. 1998. Vol. 42, N. 1. P. 24 – 28 (in Russian).

12. *Kruger A. Ya., Mordukhovich B. S.* Extremal points and the Euler equation in nonsmooth optimization problems // Dokl. Akad. Nauk BSSR. 1980. Vol. 24, N. 8. P. 684 – 687 (in Russian).

13. *Mordukhovich B. S.* Approximate Methods in Problems of Optimization and Control. Moscow: Nauka, 1988 (in Russian).

14. *Mordukhovich B. S., Shao Y.* Differential characterizations of covering, metric regularity, and Lipschitzian properties of multifunctions between Banach spaces // Nonlin. Anal. 1995. Vol. 25. P. 1401 – 1424.

15. *Mordukhovich B. S., Shao Y.* Extremal charactrizations of Asplund spaces // Proc. Amer. Math. Soc. 1996. Vol. 124. P. 197 – 205.

16. *Mordukhovich B. S., Shao Y.* Nonsmooth sequential analysis in Asplund spaces // Trans. Amer. Math. Soc. 1996. Vol. 348. P. 1235 – 1280.