$\hat{r}_i = b_i$ and $\hat{q}_i = q_i - 1$ for $r_i = 0$. For every $i \in I$ we then have $a_i = b_i\hat{q}_i + \hat{r}_i$ and hence $a = bq + r$. Moreover, for every $i \in I$ we have $\Phi_i\hat{r}_i \leqslant \Phi_i b_i$ together with $\Phi_s\hat{r}_s < \Phi_s b_s$ for at least one $s \in I$. Since the ordering in W is lexicographic we then have $(\Phi_i\hat{r}_i) < (\Phi_i b_i)$ and therefore $\Phi r < \Phi b$. Hence $a = bq + r$, $\Phi r < \Phi b$ and $r \in R(K)_0$. Consequently, Φ is a right Γ-Euclidean algorithm of the ring K.

If $I = \mathbf{N}$ and $K_i = \mathbf{R}$ $(i \in \mathbf{N})$ each of the rings K_i is Euclidean, but this is not true for the ring $K = \prod_{i \in \mathbf{N}} K_i = \mathbf{R}^{\mathbf{N}}$. In other words, in Theorem 3 the requirement "Γ-Euclidean" cannot be replaced by the condition "Euclidean" for the rings K and K_i, $i \in I$.

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PROPERTIES OF GENERALIZED DIFFERENTIALS

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The aim of the present article is to investigate the properties of generalized differentials [1] of nonsmooth and nonconvex functions in a Banach space. The above-mentioned generalized differentials are developments of the appropriate constructions from [2-4]. They are used for the formulation of necessary conditions for optimality in nonsmooth and nonconvex extremal probelms [2, 5-9].

At present there exist several different generalizations of the notions of derivative and subdifferential in the literature, among which Clarke's generalized gradient [10, 11] is most widely propagated in the theory of nonsmooth optimization. The popularity of the generalized gradient is elucidated by its successful properties, which enable us to establish the analogues of many facts of the classical differential calculus and subdifferential calculus of convex functions [10-17]. Together with them, as observed by some authors [18-20], in many cases Clarke's generalized gradient turns out to be too wide a set and the results, formulated with its help, are quite rough.

The generalized differentials, to be studied in this article, being nonconvex, are finer local characteristics of a function in comparison with Clarke's generalized gradients (and Varga's derived sets [18]). In addition, it has been possible to obtain a series of relations, analogous to the corresponding results of Clarke's theory, for them.

Many relations can be written in the form of inclusions, as for Clarke's generalized gradients. Thus, in the case of a sum of functions the fundamental formula has the form

$$D^-(\varphi_1 + \varphi_2)(x^0) \subset D^-\varphi_1(x^0) + D^-\varphi_2(x^0). \tag{1}$$

In the general case, equality does not hold in (1). This fact is characteristic for generalized differentiation (in some sense) in general classes of nonsmooth functions. The inclusion (1) enables us to decompose the elements of the generalized differential of a sum of functions in terms of elements of the generalized differentials of separate functions. By virtue of this fact, the inclusion (1) can be used for the analysis of nonsmooth functions and deduction of necessary conditions for extremum in nonsmooth optimization problems.

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In this article, we also give relations for the generalized differentials of maximum and composition of nonsmooth functions and the generalized mean-value theorem. Similar results for approximate subdifferentials have been established by Ioffe [21], who has also shown that the lower generalized differential, to be defined in Sec. 1, is the smallest set among all possible generalized derivatives that have natural properties.

Definitions of the generalized differentials and generalized normals are given in Sec. 1. The main properties of the generalized differentials are proved in Sec. 2. The inclusion (1) is proved in Sec. 3.

In this article, we use the following notation and abbreviation. We denote the topologidal adjoint of a Banach space X by X* and the norms and the unit balls in the initial and the adjoint spaces by $\|\cdot\|$, $\|\cdot\|_*$ and B, B*, respectively, and the bilinear form that defines the duality between the spaces by $\langle\cdot, \cdot\rangle$. The adjoint space is assumed to be equipped with the weak* topology. The symbols int and cl denote the interior and the closure, respectively (the closure of a set in the adjoint space is understood as the set of all weakly* limit points of bounded sequences of elements of this set). The sum of sets and product of a set by a number are defined in the following manner: $U+V=\{u+v: u\in U,\ v\in V\}$, and $\alpha U=\{\alpha u: u\in U\}$, where $\alpha \neq 0$; if $U = \emptyset$ and $\alpha = 0$, then $0\emptyset = \{0\}$. The operations over functions are defined in the usual manner. In this connection, it is assumed that $(+\infty) + (-\infty) = +\infty$ and $0 \times (\pm\infty) = 0$. The maximum of two functions φ_1 and φ_2 is denoted by $\varphi_1 \vee \varphi_2$.

1. Fundamental Definitions

Let X be a real Banach space, φ be a function from X into the extended real line $\overline{R} = [-\infty, \infty]$ that is finite at a point x^0, and ε be a nonnegative number. Let us consider the sets

$$\partial_\varepsilon^- \varphi(x^0) = \left\{x^* \in X^* : \liminf_{x\to x^0} \frac{\varphi(x) - \varphi(x^0) - \langle x^*, x - x^0\rangle}{\|x - x^0\|} \geqslant -\varepsilon\right\}, \tag{2}$$

$$D^-\varphi(x^0) = \bigcap_{\varepsilon>0} \mathrm{cl} \bigcup \left[\partial_\varepsilon^- (\mathrm{cl}^\downarrow \varphi)(x) : x \in B_\varphi^\varepsilon(x^0)\right]. \tag{3}$$

Here $\mathrm{cl}^\downarrow\varphi$ is the lower semicontinuous hull of the function φ $\left(\mathrm{cl}^\downarrow \varphi(x) = \liminf_{y\to x} \varphi(y),\quad x \in X\right)$ and $B_\varphi^\varepsilon(x^0) = \{x \in x^0 + \varepsilon B\ :\ |\mathrm{cl}^\downarrow \varphi(x) - \varphi(x^0)| = \varepsilon\}$.

The set (2) is obviously convex. It is called the lower ε-semidifferential of φ at the point x^0 (we will omit the suffix ε for $\varepsilon = 0$). If φ is Frechet-differentiable at x^0, then the set $\partial^-\varphi(x^0)$ reduces to the derivative. If φ is a convex function, then this set coincides with the subdifferential of φ at the point x^0 in the sense of convex analysis [15, 19, 22, 23]. In addition, $\partial_\varepsilon^- \varphi(x^0) = \partial^- \varphi(x^0) + \varepsilon B^*$ in both the cases.

The set (3) is called *the lower generalized semidifferential of the function* φ *at the point* x^0. In distinction from the lower ε-semidifferential (2), the lower generalized semidifferential (3) does not change if any other equivalent norm of X is used in its definition. If φ is continuous in the neighborhood of x^0, then Eq. (3) is obviously simplified:

$$D^-\varphi(x^0) = \bigcap_{\varepsilon>0} \mathrm{cl} \bigcup \left[\partial_\varepsilon^- \varphi(x)\ :\ x \in x^0 + \varepsilon B\right]. \tag{4}$$

Let us observe that the set (3) can be nonconvex. Thus, for the function $\varphi(x) = -|x|, x \in R$, we have $D^-\varphi(0) = \{-1, 1\}$.

The sets $\partial_\varepsilon^+\varphi(x^0) = -\partial_\varepsilon^-(-\varphi)(x^0)$ and

$$D^+\varphi(x^0) = -D^-(-\varphi)(x^0) \tag{5}$$

are called, respectively, *the upper* ε*-semidifferential* (the suffix ε is omitted for $\varepsilon = 0$) and *the upper generalized semidifferential of the function* φ *at the point* x^0. It is easy to write out explicit representations, analogous to Eqs. (2)-(4), for them.

The following inclusions follow from Eqs. (3) and (5):

$$D^-\varphi(x^0) \supset \partial^-\varphi(x^0), \tag{6}$$

$$D^+\varphi(x^0) \supset \partial^+\varphi(x^0). \tag{7}$$

If equality holds in the inclusion (6) [the inclusion (7)], then the function φ is said to be *lower (upper) regular at the point* x^0.

The sets (3) and (5) can be simultaneously nonempty. In addition, in distinction from Clarke's generalized gradient, they do not, in general, coincide and are not contained in one another even for Lipschitz functions (see examples in [4, 9]). The union of the two generalized semidifferentials

$$D^0\varphi(x^0)=D^-\varphi(x^0)\cup D^+\varphi(x^0) \tag{8}$$

is called *the generalized differential of the function* φ *at the point* x^0.

Certain definite local characteristics of nonconvex sets correspond to the above-described constructions of generalized semidifferentials of nonconvex functions. Let Ω be an arbitrary set in the space X and $x^0\in \mathrm{cl}\,\Omega$. Let us consider the sets

$$N_\varepsilon(x^0|\Omega)=\left\{x^*\in X^*:\limsup_{\substack{x\to x^0\\ x\in\Omega}}\frac{\langle x^*,x-x^0\rangle}{\|x-x^0\|}\leqslant\varepsilon\right\}, \tag{9}$$

$$K(x^0|\Omega)=\bigcap_{\varepsilon>0}\mathrm{cl}\bigcup[N_\varepsilon(x|\Omega):x\in\mathrm{cl}\,\Omega\cap(x^0+\varepsilon B)]. \tag{10}$$

The set (9), the set of ε-normals to the set Ω at the point x^0, is convex. For $\varepsilon = 0$ (we omit the suffix ε in this case) it represents a cone, which is called *the normal cone*. In the case of the convex set Ω, as is easily observed, it coincides with the normal cone in the sense of convex analysis. The set (10) is also a cone, possibly nonconvex. It is called *the generalized normal cone to the set* Ω *at the point* x^0. The elements of the cone (10) are called the *generalized normals*.

It follows from definitions (9) and (10) that

$$N_\varepsilon(x^0|\Omega)=N_\varepsilon(x^0|\mathrm{cl}\,\Omega),\quad K(x^0|\Omega)=K(x^0|\mathrm{cl}\,\Omega), \tag{11}$$

$$K(x^0|\Omega)\supset N(x^0|\Omega). \tag{12}$$

If equality holds in (12), then we will say that *the set* Ω *is regular at the point* x^0.

Let us observe that the sets (9) and (10) coincide, respectively, with the lower ε-semidifferential and the lower generalized semidifferential (or the generalized differential) of the indicator function† of the set $\Omega\cup\{x^0\}$ at the point x^0. Thus, the notion of the generalized normal cone may be considered as derived from the notion of the lower generalized semidifferential. At the same time, as the following theorem shows, the reverse connection also exists between these notions: The lower generalized semidifferential of the function φ can be defined in terms of generalized normals to its epigraph $E(\varphi)=\{(x,\mu)\in X\times R:\varphi(x)\leqslant\mu\}$ or to its graph $G(\varphi)=\{(x,\mu)\in X\times R:\varphi(x)=\mu\}$.

THEOREM 1. The following representation is valid:

$$D^-\varphi(x^0)=\{x^*\in X^*:(x^*,-1)\in K(x^0,\varphi(x^0)|E(\varphi))\}. \tag{13}$$

If φ is lower semicontinuous in the neighborhood of the point x^0, then the set $E(\varphi)$ in Eq. (13) can be replaced by the set $G(\varphi)$.

Proof. Let $x^*\in D^-\varphi(x^0)$. By definition, there exist sequences $\{x_k\}\subset X$, $\{x_k^*\}\subset X^*$, and $\{\varepsilon_k\}\subset R_+$ such that $x_k^*\in\partial^-_{\varepsilon_k}(\mathrm{cl}^\downarrow\varphi)(x_k)$ for k = 1, 2,..., and $x_k\to x^0$, $\mathrm{cl}^\downarrow\varphi(x_k)\to\varphi(x^0)$, and $\varepsilon_k\to 0$ as $k\to\infty$, the sequence $\{x_k^*\}$ is bounded, and x^* is a limit point of it. Since $\|x-x_k\|\leqslant\|(x,\mu)-(x_k,\mathrm{cl}^\downarrow\varphi(x_k))\|$ for all $(x,\mu)\in X\times R$, it follows from the definitions (2) and (9) that $(x_k^*,-1)\in N_{\varepsilon_k}(x_k,\mathrm{cl}^\downarrow\varphi(x_k)|\mathrm{cl}\,E(\varphi))$, for k = 1, 2,...; whence, by virtue of (11),

$$(x^*,-1)\in K(x^0,\varphi(x^0)|E(\varphi)). \tag{14}$$

Conversely, let condition (14) be fulfilled. Then there exist sequences $\{x_k,\ \mu_k\}\subset\mathrm{cl}\,E(\varphi)$, $\{x_k^*,\lambda_k\}\subset X^*\times R$, and $\{\varepsilon_k\}\subset R_+$ such that $(x_k^*,\lambda_k)\in N_{\varepsilon_k}(x_k,\mu_k|E(\varphi))$ for k = 1, 2,..., $(x_k,\mu_k)\to(x^0,\varphi(x^0))$ and $\varepsilon_k\to 0$ as $k\to\infty$, the sequence $\{x_k^*,\lambda_k\}$ is bounded, and $(x^*,-1)$ is a limit point of it. Without loss of generality, we will assume that $\varepsilon_k<1$ and $\lambda_k=-1$ for k = 1, 2,... . Then it follows from the definition of the ε-normal that $\mu_k=\mathrm{cl}^\downarrow\varphi(x_k)$ for k = 1, 2,... . We show that $x_k^*\in\partial^-_{\hat\varepsilon_k}(\mathrm{cl}^\downarrow\varphi)(x_k)$, where $\hat\varepsilon_k=(1+\|x_k^*\|_*)(1-\varepsilon_k)^{-1}\varepsilon_k$. Since $\hat\varepsilon_k\to 0$ as $k\to\infty$, it follows from this condition that $x^*\in D^-\varphi(x^0)$.

Let us assume the contrary: There exist a sequence $\{y_i\}\subset X$ that converges to x_k and a number $\varepsilon>\hat\varepsilon_k$ such that

†The indicator function of a set Ω is defined as follows: $\delta(x|\Omega)=0$ if $x\in\Omega$ and $\delta(x|\Omega)=\infty$ if $x\notin\Omega$.

$$\mathrm{cl}^{\downarrow}\varphi(y_i)-\mathrm{cl}^{\downarrow}\varphi(x_k)-\langle x_k^*, y_i-x_k\rangle+\varepsilon\|y_i-x_k\|<0, \quad i=1,2,\ldots.$$

Let us set $\mu_i=\mathrm{cl}^{\downarrow}\varphi(x_k)+\langle x_k^*, y_i-x_k\rangle-\varepsilon\|y_i-x_k\|$. It follows from the last inequality that $(y_i,\mu_i)\in \mathrm{cl}\,E(\varphi)$. In addition, it is obvious that $\mu_i\to \mathrm{cl}^{\downarrow}\varphi(x_k)$ as $i\to\infty$. We have

$$\frac{\langle x_k^*, y_i-x_k\rangle-(\mu_i-\mathrm{cl}^{\downarrow}\varphi(x_k))}{\|(y_i,\mu_i)-(x_k,\mathrm{cl}^{\downarrow}\varphi(x_k))\|}\geqslant\frac{\varepsilon}{1+\|x_k^*\|_*+\varepsilon}>\varepsilon_k, \quad i=1,2,\ldots,$$

i.e., $(x_k^*,-1)\notin N_{\varepsilon_k}(x_k,\mathrm{cl}^{\downarrow}\varphi(x_k)\,|\,E(\varphi))$. This contradiction completes the proof of Eq. (13).

To prove the second part of the theorem, it is sufficient to observe that the condition $(x^*,-1)\in K(x^0,\varphi(x^0)\,|\,E(\varphi))$ implies the condition $(x^*,-1)\in K(x^0,\varphi(x^0)\,|\,G(\varphi))$, and if the function φ is lower semicontinuous at a point x, then it follows from the condition $(x^*,-1)\in N_\varepsilon(x,\varphi(x)\,|\,G(\varphi))$ that $(x^*,-1)\in N_\varepsilon(x,\varphi(x)\,|\,E(\varphi))$. The theorem is proved.

Thus, the lower generalized semidifferential of the function φ is defined by "nonhorizontal" generalized normals to its epigraph.

Let us observe that actually the following analogous representation for the lower semidifferential has also been established in the proof of Theorem 1:

$$\partial^-\varphi(x^0)=\{x^*\in X^*:(x^*,-1)\in N(x^0,\varphi(x^0)\,|\,E(\varphi))\}. \tag{15}$$

In its turn, the generalized normal cone to the epigraph of the function φ at the point $(x^0,\varphi(x^0))$ is completely determined by the set (13) and the set

$$D_s^-\varphi(x^0)=\{x^*\in X^*:(x^*,0)\in K(x^0,\varphi(x^0)\,|\,E(\varphi))\}. \tag{16}$$

Following Rockafellar [24], we call the set (16) *the singular lower generalized semidifferential.*

From Theorem 1 we easily deduce an equivalent representation for the upper generalized semidifferential of a function in terms of the generalized normals to a subgraph of it.

Let us now suppose that φ is a continuous function that takes values in a Banach space Z. For each $z^*\in Z^*$, let us define a scalar-valued function $L_{z^*}(x)=\langle z^*,\varphi(x)\rangle$, $x\in X$, and introduce the sets

$$D\varphi(x^0)=\bigcap_{\varepsilon>0}\mathrm{cl}\bigcup\{(x^*,z^*)\in X^*\times Z^*:x^*\in\partial_\varepsilon^-L_{z^*}(x),\ x\in x^0+\varepsilon B\},$$
$$D_{z^*}\varphi(x^0)=\{x^*\in X^*:(x^*,z^*)\in D\varphi(x^0)\}. \tag{17}$$

The set (17) is called *the generalized differential of the function* φ *at the point* x^0 *in the direction* z^*.

In the sequel, speaking about the totality of the constructions (3), (5), (8), (16), and (17), we will call them *generalized differentials.*

2. Properties of Generalized Differentials

The following two theorems establish a connection between the above-introduced constructions and the classical notions of a derivative and a subdifferential.

THEOREM 2. If a function $\varphi:X\to R$ is strongly differentiable† at a point x^0, then

$$D^0\varphi(x^0)=D^-\varphi(x^0)=D^+\varphi(x^0)=\{\varphi'(x^0)\}. \tag{18}$$

Proof. Let the function φ be strongly differentiable at x^0. Then it is continuous in the neighborhood of this point and for $\delta>0$ there exists a positive number ε such that $\partial_\varepsilon^-\varphi(x)\subset\varphi'(x^0)+\delta B^*$ for all $x\in x^0+\varepsilon B$. It follows from Eq. (4) that $D^-\varphi(x^0)=\{\varphi'(x^0)\}$. The remaining equalities in (18) are the consequences of what we have written by virtue of the definitions (5) and (8).

THEOREM 3. If $\varphi:X\to\bar{R}$ is a convex function that is finite at a point x^0, then

$$D^0\varphi(x^0)=D^-\varphi(x^0)=\partial^-\varphi(x^0). \tag{19}$$

†A function $\varphi:X\to Z$ is said to be *strongly differentiable* [25] *at a point* x^0 [with the derivative $\varphi'(x^0)$] if

$$\lim_{x\to x^0,y\to x^0}\frac{\varphi(x)-\varphi(y)-\varphi'(x^0)(x-y)}{\|x-y\|}=0.$$

Proof. Let φ be a convex function and $x^* \in D^-\varphi(x^0)$. By definition, there exist sequences $\{x_k\} \subset X$, $\{x_k^*\} \subset X^*$, and $\{\varepsilon_k\} \subset R_+$ such that

$$x_k^* \in \partial_{\varepsilon_k}^-(\mathrm{cl}^\downarrow \varphi)(x_k), \quad k = 1, 2, \ldots, \quad x_k \to x^0, \quad \mathrm{cl}^\downarrow \varphi(x_k) \to \varphi(x^0),$$

$\varepsilon_k \to 0$ as $k \to \infty$, the sequence $\{x_k^*\}$ is bounded, and x^* is a limit point of it. By virtue of the convexity of the $\mathrm{cl}^\downarrow \varphi$, for each k there exists a point $y_k^* \in x_k^* + \varepsilon_k B^*$ such that $y_k^* \in \partial^- \times (\mathrm{cl}^\downarrow \varphi)(x_k)$, i.e.,

$$\mathrm{cl}^\downarrow \varphi(x) - \mathrm{cl}^\downarrow \varphi(x_k) \geqslant \langle y_k^*, x - x_k \rangle \tag{20}$$

for all $x \in X$. Let x be an arbitrary point of the space X. Without loss of generality, we can assume that $\langle x_k^* - x^*, x - x^0 \rangle \to 0$ as $k \to \infty$. In this case, it is obvious that $\langle y_k^* - x^*, x - x_k \rangle \to 0$ and, taking limit in (20), we conclude that $\varphi(x) - \varphi(x^0) \geqslant \langle x^*, x - x^0 \rangle$ for all $x \in X$, i.e., $x^* \in \partial^-\varphi(x^0)$. By virtue of (6), we have $D^-\varphi(x^0) = \partial^-\varphi(x^0)$.

If $\partial_\varepsilon^-(\mathrm{cl}^\downarrow(-\varphi))(x) \neq \varnothing$ for certain $x \in X$ and $\varepsilon > 0$, then the function φ is bounded above in the neighborhood of x and is, consequently, [22, 23] continuous and subdifferentiable at this point: $\partial^-\varphi(x) \neq \varnothing$. Therefore, $\partial_\varepsilon^+\varphi(x) = -\partial_\varepsilon^-(-\varphi)(x) \subset \partial^-\varphi(x) + \varepsilon B^*$. By virtue of the upper-semicontinuity of the subdifferential mapping of a convex function [22], it follows from the last inclusion that $D^+\varphi(x^0) \subset D^-\varphi(x^0)$. The theorem is proved.

COROLLARY. If Ω is a convex set and $x^0 \in \mathrm{cl}\,\Omega$, then $K(x^0|\Omega) = N(x^0|\Omega)$.

Thus, each function that is strongly differentiable at a point is lower as well as upper regular at this point, convex functions are lower regular at all points at which they take finite values, and convex sets are regular at all points of their closures.

If a function $\varphi: X \to R$ satisfies the Lipschitz condition in the neighborhood of a point x^0, then it is easily seen that the set $D^0\varphi(x^0)$ is bounded. In addition, $D_s^-\varphi(x^0) = \{0\}$.

The following two theorems follow immediately from definitions.

THEOREM 4. If x^0 is a point of local minimum of a function $\varphi: X \to \bar{R}$, then $0 \in D^-\varphi(x^0)$.

THEOREM 5. If a function $\varphi: X \to \bar{R}$ is finite at a point x^0, then $D^-\varphi(x^0) = \bigcap_{\varepsilon > 0} \mathrm{cl} \bigcup [D^-(\mathrm{cl}^\downarrow \varphi)(x) : x \in B_\varphi^\varepsilon(x^0)]$.

If φ is continuous in the neighborhood of x^0, then the assertion of Theorem 5 means that the multivalued mapping $D^-\varphi(\cdot)$ is upper-semicontinuous at x^0.

COROLLARY. If $x^0 \in \mathrm{cl}\,\Omega$, then

$$K(x^0|\Omega) = \bigcap_{\varepsilon > 0} \mathrm{cl} \bigcup [K(x|\Omega) : x \in \mathrm{cl}\,\Omega \cap (x^0 + \varepsilon B)].$$

In the case of a finite-dimensional space X, the definition (10) of the generalized normal cone is simplfied: The sets of ε-normals can be replaced by normal cones.

THEOREM 6. Let $x^0 \in \mathrm{cl}\,\Omega$. If $\dim X < \infty$, then

$$K(x^0|\Omega) = \bigcap_{\varepsilon > 0} \mathrm{cl} \bigcup [N(x|\Omega) : x \in \mathrm{cl}\,\Omega \cap (x^0 + \varepsilon B)].$$

Proof. Let $\dim X < \infty$. Without loss of generality, we will assume the space X to be Euclidean and identity the space X^* with X. Let $x^* \in K(x^0|\Omega) \setminus \{0\}$. Then there exist sequences $\{x_k\} \subset \mathrm{cl}\,\Omega$, $\{x_k^*\} \subset X^*$, and $\{\varepsilon_k\} \subset R_+ \setminus \{0\}$ such that $x_k \to x^0$, $x_k^* \to x^*$, and $\varepsilon_k \to 0$ as $k \to \infty$ and $x_k^* \subset N_{\varepsilon_k}(x_k|\Omega)$ for $k = 1, 2, \ldots$. For each k let us consider the ray $y_k(\alpha) = x_k + \alpha x_k^*$, $\alpha \geqslant 0$. Let $x_k(\alpha)$ denote the point of $\mathrm{cl}\,\Omega$ that is nearest to $y_k(\alpha)$. It is obvious that $x_k(\alpha) \to x_k$ as $\alpha \to 0$. By the definition of the ε-normal, we can find a sufficiently small number $\alpha_k \in (0, 1)$, such that $\langle x_k^*, \bar{x}_k - x_k \rangle \leqslant 2\varepsilon_k \|\bar{x}_k - x_k\|$, where $\bar{x}_k = x_k(\alpha_k)$, $\bar{y}_k = y_k(\alpha_k)$. We have

$$\|\bar{x}_k - x_k\|^2 = \|\bar{y}_k - \bar{x}_k\|^2 - \|\bar{y}_k - x_k\|^2 + 2\alpha_k \langle x_k^*, \bar{x}_k - x_k \rangle \leqslant 4\alpha_k \varepsilon_k \|\bar{x}_k - x_k\|,$$

whence $\|\bar{x}_k - x_k\| \leqslant 4\alpha_k \varepsilon_k$. Let us set $\bar{x}_k^* = x_k^* + \alpha_k^{-1}(x_k - \bar{x}_k)$. It follows from the last inequality that $\|\bar{x}_k^* - x_k^*\| \leqslant 4\varepsilon_k$. Thus, $\bar{x} \to x^0$ and $\bar{x}_k^* \to x^*$ as $k \to \infty$. In addition, for each $x \in \Omega$ we have

$$\begin{aligned} 2\alpha_k \langle \bar{x}_k^*, x - \bar{x}_k \rangle &= \langle \bar{y}_k - \bar{x}_k, x - \bar{x}_k \rangle + \langle \bar{y}_k - x, x - \bar{x}_k \rangle + \|x - \bar{x}_k\|^2 \\ &= \|\bar{y}_k - \bar{x}_k\|^2 - \|\bar{y}_k - x\|^2 + \|x - \bar{x}_k\|^2 \leqslant \|x - \bar{x}_k\|^2, \end{aligned}$$

whence it follows that $\bar{x}_k^* \in N(\bar{x}_k|\Omega)$. The theorem is proved.

By virtue of Eqs. (13) and (15), an equivalent representation for the lower generalized semidifferential, characteristic for a finite-dimensional space, follows from Theorem 6.

COROLLARY. Let a function $\varphi: X \to \bar{R}$ be finite at a point x^0. If $\dim X < \infty$, then

$$D^{-}\varphi(x^0) = \bigcap_{\varepsilon>0} \operatorname{cl} \bigcup \left[\partial^{-}(\operatorname{cl}^{\downarrow}\varphi)(x) : x \in B_{\varphi}^{\varepsilon}(x^0)\right].$$

If a function $\varphi: X \to \bar{R}$ is finite at a point x^0, then we can define the following set for it:

$$\partial_C\varphi(x^0) = \{x^* \in X^* : \varphi^{\uparrow}(x^0; x) \geqslant \langle x^*, x\rangle \text{ for all } x \in X\}, \tag{21}$$

where

$$\varphi^{\uparrow}(x^0; x) = \sup_{\alpha>0} \inf_{\delta>0} \sup_{\substack{\|(y,\mu)-(x^0,\varphi(x^0))\|\leqslant\delta \\ \lambda>0\ \mu\geqslant\varphi(y),\ t\in(0,\lambda)}} \inf_{v\in x+\alpha B} \frac{\varphi(y+tv)-\mu}{t} \tag{22}$$

is the upper subderivative [14, 15] of φ at the point x^0 in the direction x. The set (21) is called *Clarke's generalized gradient of the function* φ *at the point* x^0. The function $\varphi^{\uparrow}(x^0; \cdot)$ is a positively homogeneous lower semicontinuous convex function. If φ is directionally Lipschitz [14, 15] at a point x^0 in the direction x, i.e.,

$$\limsup_{\substack{t\to+0,\ v\to x \\ (y,\mu)\to(x^0,\varphi(x^0)) \\ \mu\geqslant\varphi(y)}} \frac{\varphi(y+tv)-\mu}{t} < \infty, \tag{23}$$

then the upper subderivative (22) coincides with Clarke's generalized derivative [10, 11]

$$\varphi^0(x^0; x) = \limsup_{t\to+0,\ y\to x^0} \frac{\varphi(y+tx)-\varphi(y)}{t}.$$

In addition, the set of those x, for which the condition (23) is fulfilled, coincides with the interior of the effective domain $\operatorname{dom}\varphi^{\uparrow}(x^0; \cdot)$ of the function $\varphi^{\uparrow}(x^0; \cdot)$.

If $x^0 \in \operatorname{cl}\Omega$, then we can introduce the cones

$$\begin{aligned} T_C(x^0|\Omega) = \{x \in X :\ & \text{for each } \varepsilon > 0 \text{ there exists} \\ & \text{a } \delta > 0, \text{ such that } [y + t(x + \varepsilon B)] \cap \Omega \neq \varnothing, \\ & \text{provided } y \in \Omega \cap (x^0 + \delta B),\ t \in (0, \delta)\}, \end{aligned} \tag{24}$$

$$N_C(x^0|\Omega) = \{x^* \in X^* : \langle x^*, x\rangle \leqslant 0 \text{ for all } x \in T_C(x^0|\Omega)\}, \tag{25}$$

which are called, respectively, *Clarke's tangent and normal cones to the set* Ω *at the point* x^0 [10, 11, 14, 15].

Let us observe that the set (24) coincides with the effective domain of the upper subderivative of the indicator function of the set $\Omega \cup \{x^0\}$ at the point x^0 and the set (25) coincides with Clarke's generalized gradient of it at this point. The above indicator function is directionally Lipschitz at x^0 in the direction x if and only if the set Ω is epi-Lipschitz at x^0 in the direction x, i.e., there exists a number $\delta > 0$ such that $y + tv \in \Omega$ for all $y \in \Omega \cap (x^0 + \delta B)$, $t \in (0, \delta)$, and $v \in x^0 + \delta B$. In addition, the set of all these x coincides with the interior of the set (24).

The properties of Clarke's generalized gradients and normal cones are set forth in [10, 11, 14, 15]. They are similar in many ways to the corresponding properties of the above-described generalized differentials and generalized normal cones. However, in distinction from the latter ones, the sets (21) and (25) are always convex. The following theorem establishes a relation between Clarke's generalized gradient and the lower generalized semidifferential.

THEOREM 7. If a function $\varphi: X \to \bar{R}$ is finite at a point x^0, then

$$D^{-}\varphi(x^0) \subset \partial_C\varphi(x^0).$$

Proof. $x \in X$ and $x^* \in D^{-}\varphi(x^0)$. To prove the theorem, it is sufficient to show that $\langle x^*, x\rangle \leqslant \varphi^{\uparrow}(x^0; x)$. By the definition of the lower generalized semidifferential, there exist sequences $\{x_k\} \subset X$, $\{x_k^*\} \subset X^*$, and $\{\varepsilon_k\} \subset R_+$, such that $x_k^* \in \partial_{\varepsilon_k}^{-}(\operatorname{cl}^{\downarrow}\varphi)(x_k)$, for k = 1, 2,..., $x_k \to x^0$, $\operatorname{cl}^{\downarrow}\varphi(x_k) \to \varphi(x^0)$, $\varepsilon_k \to 0$ as $k \to \infty$, the sequence $\{x_k^*\}$ is bounded, and x^* is a limit point of it. Passing, if necessary, to subsequences, we assume that $\langle x_k^* - x^*, x\rangle \to 0$ as $k \to \infty$. It follows from Eq. (22) that for each k there exists a number $\delta_k > 0$ such that

$$\inf_{v\in x+\frac{1}{k}B} \frac{\mathrm{cl}^{\downarrow}\varphi(y+tv)-\mu}{t} \leqslant \varphi^{\uparrow}(x^0; x)+\frac{1}{k}$$

for all $t\in(0,\delta_k)$ and $(y,\mu)\in \mathrm{cl}\,E(\varphi)\cap[(x^0,\varphi(x^0))+\delta_k B]$. Without loss of generality, we can assume that $\|(x_k, \mathrm{cl}^{\downarrow}\varphi(x_k))-(x^0,\varphi(x^0))\|\leqslant\delta_k$. Then for each $t\in(0,\delta_k)$ there exists a point $v_t\in x+B/k$ such that

$$\frac{\mathrm{cl}^{\downarrow}\varphi(x_k+tv_t)-\mathrm{cl}^{\downarrow}\varphi(x_k)}{t}\leqslant\varphi^{\uparrow}(x^0;x)+\frac{2}{k}.$$

By virtue of the definition of the lower ε-semidifferential, from the last inequality we get

$$\varphi^{\uparrow}(x^0;x)-\langle x_k^*, x\rangle \geqslant -\varepsilon_k\left(\|x\|+\frac{1}{k}\right)-\frac{1}{k}\|x_k^*\|_*-\frac{2}{k},$$

whence the desired result follows on taking limit as $k \to \infty$.

COROLLARY 1. If $x^0\in \mathrm{cl}\,\Omega$, then $K(x^0|\Omega)\subset N_C(x^0|\Omega)$.

By virtue of the known properties of the generalized gradients of Lipschitz functions [10, 11], the following result also follows from Theorem 7.

COROLLARY 2. If a function $\varphi: X\to R$ satisfies the Lipschitz condition in the neighborhood of a point x^0, then

$$D^0\varphi(x^0)\subset\partial_C\varphi(x^0). \tag{26}$$

Let us observe that for non-Lipschitz functions the inclusion (26) cannot be fulfilled. At the same time, it follows from Theorem 7 that the generalized differential $D^0\varphi(x^0)$ is always contained in the symmetric generalized gradient [17] of the function φ at the point x^0.

In the conclusion of the section, we give a result, which characterizes the generalized differentials of the functions that take values in a Banach space.

THEOREM 8. If a function $\varphi: X\to Z$ is continuous in the neighborhood of a point x^0, then for arbitrary $z^*\in Z^*$

$$D_{z^*}\varphi(x^0)\supset D^{-}L_{z^*}(x^0), \tag{27}$$

where $L_{z^*}(x)=\langle z^*, \varphi(x)\rangle$, $x\in X$. Each of the following conditions is sufficient for the fulfillment of equality in (27):

(a) φ is strongly differentiable at x^0.

(b) $\dim Z < \infty$ and φ satisfies the Lipschitz condition in the neighborhood of x^0.

3. A Theorem on the Lower Generalized Semidifferential of a Sum of Functions

In this section, we establish the conditions, under which the inclusion (1), the fundamental relation that characterizes the generalized differentials of nonsmooth functions, is valid. Everywhere in the sequel, we assume that the spaces X and Z satisfy the following condition.

Condition A. There exists a scalar-valued function ψ on the space such that $\psi(0) = 0$ and $\psi(x) > 0$ for $x \neq 0$; it is continuous in a neighborhood U of the origin and is Frechet-differentiable in $U\setminus\{0\}$; in addition, $\|\psi'(x)\|_*\geqslant 1$ for $x\in U\setminus\{0\}$.

Condition A is fulfilled, e.g., in each reflexive space.

THEOREM 9. Let the function $\varphi_1: X\to\bar{R}$ and $\varphi_2: X\to\bar{R}$ be finite at a point x^0 and lower semicontinuous in its neighborhood. If the function φ_2 is directionally Lipschitz at x^0 (in a certain direction) and

$$\mathrm{dom}\,\varphi_1^{\uparrow}(x^0;\cdot)\cap\mathrm{int}\,\mathrm{dom}\,\varphi_2^{\uparrow}(x^0;\cdot)\neq\varnothing, \tag{28}$$

then the condition (1) is fulfilled. If, in addition, the functions φ_1 and φ_2 are lower regular at x^0, then the function $\varphi_1+\varphi_2$ is also lower regular at this point and equality holds in (1).

By virtue of Theorems 2 and 3, the above theorem generalizes the classical theorems of differential calculus and convex analysis. The requirement for the function φ_2 to be directionally Lipschitz, in conjunction with the condition (28), is a natural generalization of the corresponding condition in the formulation of the Moreau–Rockafellar theorem [23]. If the function φ_2 satisfies the Lipschitz condition in the neighborhood of x^0, then it is directionally Lipschitz at this point and $\operatorname{dom}\varphi_2^{\uparrow}(x^0;\cdot)=X$ [14]. In this case, the following result follows from Theorem 9.

COROLLARY 1. Let a function $\varphi_1: X\to\bar{R}$ be finite at a point x^0 and lower semicontinuous in its neighborhood and a function $\varphi_2: X\to R$ satisfy the Lipschitz condition in the neighborhood of x^0. Then the condition (1) is fulfilled and

$$D^-(\varphi_1+\varphi_2)(x^0)-D^+\varphi_2(x^0)\supset D^-\varphi_1(x^0). \tag{29}$$

The condition (29) is obtained as a result of application of Theorem 9 to the functions $\varphi_1+\varphi_2$ and $-\varphi_2$.

COROLLARY 2. Let a function $\varphi_1: X\to\bar{R}$ be finite at a point x^0 and lower semicontinuous in its neighborhood and the function φ_2 be strongly differential be at x^0. Then

$$D^-(\varphi_1+\varphi_2)(x^0)=D^-\varphi_1(x^0)+\varphi_2'(x^0).$$

We can show (see [1]) that the assertions of Corollaries 1 and 2 are valid without the assumption of the lower semicontinuity of φ_1.

COROLLARY 3. Let Ω_1 and Ω_2 be closed sets in the space X and $x^0\in\Omega_1\cap\Omega_2$. If the set Ω_2 is epi-Lipschitz at the point x^0 and $T_C(x^0|\Omega_1)\cap\operatorname{int}T_C(x^0|\Omega_2)\neq\varnothing$, then

$$K(x^0|\Omega_1\cap\Omega_2)\subset K(x^0|\Omega_1)+K(x^0|\Omega_2). \tag{30}$$

If, in addition, the sets Ω_1 and Ω_2 are regular at x^0, then the set $\Omega_1\cap\Omega_2$ is also regular at this point and equality holds in (30).

Since, as observed above, the generalized differentials are, in general, nonconvex, we cannot use the apparatus of convex analysis for their analysis. The generalized Euler equation [1, 7] lies at the basis of the proof of Theorem 9. This result characterizes the extremal disposition of a system of sets in the space.

Let $\Omega_1, \Omega_2,\ldots,\Omega_n$ be closed sets in a space X. A system of sets Ω_i, i = 1, 2,...,n, is said to be *extremal* if the following conditions are fulfilled: 1) $\bigcap_{i=1}^{n}\Omega_i\neq\varnothing$; 2) There exist sequences $\{a_k^i\}\subset X$, i = 1, 2,...,n, that converge to zero and are such that $\bigcap_{i=1}^{n}\Omega_{ik}=\varnothing$, where $\Omega_{ik}=\{x-a_k^i: x\in\Omega_i\}$, for i = 1, 2,...,n and k = 1, 2,... . In addition, each point $x^0\in\bigcap_{i=1}^{n}\Omega_i$ is called *an extremal point of the system of sets* Ω_i, i = 1, 2,...,n.

THEOREM 10. Let x^0 be an extremal point of the system of closed sets Ω_i, i = 1, 2,..., n. Then for each $\varepsilon > 0$ there exist points $y_i\in\Omega_i\cap(x^0+\varepsilon B)$ and $x_i^*\in N_\varepsilon(y_i|\Omega_i)$, i = 1, 2,...,n, such that $\sum_{i=1}^{n}\|x_i^*\|_*=1$ and

$$x_1^*+x_2^*+\ldots+x_n^*=0. \tag{31}$$

If, in addition, the sets Ω_i, i = 1, 2,...,n − 1, are eip-Lipschitz at a point x^0, then there exist elements $x_i^*\in K(x^0|\Omega_i)$, i = 1, 2,...,n, that are not simultaneously equal to zero and are are such that the condition (31) is fulfilled.

This theorem is a generalization of the theorems of Dubovitskii and Milyutin [26] and Boltyanskii [27] to the case of nonconvex sets. In analogy with [26], the relation (31) is called *the generalized Euler equation*. Theorem 10 (in a somewhat more general formulation) is proved in [1].

Proof of Theorem 8. Let $x^*\in D^-(\varphi_1+\varphi_2)(x^0)$. Then there exist sequences $\{x_k\}\subset X$, $\{x_k^*\}\subset X^*$, and $\{\varepsilon_k\}\subset R_+\setminus\{0\}$, such that $x_k^*\in\partial_{\varepsilon_k}^-(\varphi_1+\varphi_2)(x_k)$ for k = 1, 2,..., and $x_k\to x^0$, $\varphi_i(x_k)\to\varphi_i(x^0)$, i = 1, 2, and $\varepsilon_k\to 0$ as $k\to\infty$, the sequence $\{x_k^*\}$ is bounded, and x^* is a limit point of it. For each k we can find a neighborhood U of x_k such that

$$(\varphi_1+\varphi_2)(x)-(\varphi_1+\varphi_2)(x_k)-\langle x_k^*, x-x_k\rangle+2\varepsilon_k\|x-x_k\|\geqslant 0$$

for all $x \in U$. Without loss of generality, we can assume that the functions φ_1 and φ_2 are lower semicontinuous in U. Let us consider the following closed sets in the space X × R:

$$\Omega_1 = \{(x, \mu) \in U \times R : \varphi_1(x) - \varphi_1(x_k) \leqslant \mu\},$$
$$\Omega_2 = \{(x, \mu) \in U \times R : \varphi_2(x) - \varphi_2(x_k) - \langle x_k^*, x - x_k\rangle + 2\varepsilon_k \|x - x_k\| \leqslant -\mu\}.$$

They obviously form an extremal system. In addition, $(x_k, 0) \in \Omega_1 \cap \Omega_2$. By virtue of Theorem 10, there exist points $(x_{ik}, \mu_{ik}) \in E(\varphi_i) \cap [(x_{ik}, \varphi_i(x_k)) + \varepsilon_k B]$, i = 1, 2, and an element $(\hat{x}_k^*, \lambda_k) \in X^* \times R$ such that $\|\hat{x}_k^*, \lambda_k\|_* = 1$,

$$(x_k^*, \lambda_k) \in N_{\varepsilon_k}(x_{1k}, \mu_{1k} - \varphi_1(x_k) \mid \Omega_1), \quad -(x_k^*, \lambda_k) \in N_{\varepsilon_k}(x_{2k}, -\mu_{2k} + \varphi_2(x_k) + \langle x_k^*, x_{2k} - x_k\rangle - 2\varepsilon_k \|x_{2k} - x_k\| \mid \Omega_2).$$

Thus, $(x_{ik}, \mu_{ik}) \to (x^0, \varphi_i(x^0))$, i = 1, 2, as $k \to \infty$. Let $(\hat{x}^*, \lambda)$ denote a limit point of the sequence $\{x_k^*, \lambda_k\}$. Then $\lambda \leqslant 0$ and $(\hat{x}^*, \lambda) \in K(x^0, \varphi_1(x^0) \mid E(\varphi_1))$. By the definition of the ε-normal, for each k we have

$$\limsup_{\substack{(x,\mu)\to(x_{2k},\mu_{2k})\\ \mu\geqslant\varphi_2(x)}} \frac{-\langle \hat{x}_k^*, x - x_{2k}\rangle + \lambda_k(\mu - \mu_{2k} - \langle x_k^*, x - x_{2k}\rangle + 2\varepsilon_k\|x - x_{2k}\|)}{\|(x - x_{2k}, \mu - \mu_{2k} - \langle x_k^*, x - x_{2k}\rangle + 2\varepsilon_k\|x - x_{2k}\|)\|} \leqslant \varepsilon_k,$$

whence it follows that $(-\lambda_k x_k^* - \hat{x}_k^*, \lambda_k) \in N_{\hat{\varepsilon}_k}(x_{2k}, \mu_{2k} \mid E(\varphi_2))$, where $\hat{\varepsilon}_k = \varepsilon_k(1 + \|x_k^*\|_* + 2\varepsilon_k + 2|\lambda_k|)$. Since $\hat{\varepsilon}_k \to 0$ as $k \to \infty$ we have $(-\lambda x^* - \hat{x}^*, \lambda) \in K(x^0, \varphi_2(x^0) \mid E(\varphi_2))$.

We show that $\lambda \neq 0$. Since the function φ_2 is directionally Lipschitz, for each point $x \in \operatorname{int}\operatorname{dom}\varphi_2^{\uparrow}(x^0; \cdot)$ there exist numbers $C > 0$ and $\delta > 0$ such that $\varphi_2(y + tv) \leqslant \mu + Ct$ for all $(y, \mu) \in E(\varphi_2) \cap [(x^0, \varphi_2(x^0)) + \delta B]$, $t \in (0, \delta)$, and $v \in x + \delta B$. Then for all $v \in x + \delta B$ and all sufficiently large k we have

$$-\langle \lambda_k x_k^* + \hat{x}_k^*, v\rangle + \lambda_k C \leqslant \hat{\varepsilon}_k(\|v\| + C),$$

whence

$$\langle \lambda_k x_k^* + \hat{x}_k^*, x\rangle \geqslant \delta\|\lambda_k x_k^* + \hat{x}_k^*\|_* + \lambda_k C - \hat{\varepsilon}_k(\|x\| + C + \delta). \tag{32}$$

Without loss of generality, we can assume that $\langle \hat{x}_k^*, x^*, x\rangle \to 0$ as $k \to \infty$. Now if it is assumed that $\lambda_k \to 0$, then $\|\hat{x}_k^*\| \to 1$ and it follows from (32) that $\langle \hat{x}^*, x\rangle \geqslant \delta$. On the other hand, by virtue of Corollary 1 of Theorem 7 and the representation of Clarke's tangent cone to the epigraph [14, 15], $\langle \hat{x}^*, x\rangle \leqslant 0$ for all $x \in \operatorname{dom}\varphi_1^{\uparrow}(x^0; \cdot)$. The obtained inequalities contradicts the condition (28). Thus, $\lambda \neq 0$, and it follows from Theorem 1 that $x_1^* \in D^-\varphi_1(x^0)$ and $x - x_1^* \in D^-\varphi_2(x^0)$, where $x_1^* = \hat{x}^*/|\lambda|$. The inclusion (1) is proved.

If the functions φ_1 and φ_2 are lower regular at a point x^0, then $D^-(\varphi_1 + \varphi_2)(x^0) \subset \partial^-(\varphi_1 + \varphi_2)(x^0) = D^-\varphi_1(x^0) + D^-\varphi_2(x^0)$; whence it follows that the function $\varphi_1 + \varphi_2$ is lower regular at the point x^0 and equality holds in (1).

Relying on Theorem 9 (and Corollary 3 of it), we can obtain a series of other relations that characterize the generalized differential of nonsmooth functions. The proofs of the following Theorems 11 and 12 are given in [1].

THEOREM 11. Let the functions $\varphi_1 : X \to \bar{R}$ and $\varphi_2 : X \to \bar{R}$ be lower semicontinuous in the neighborhood of a point x^0, $\varphi_1(x^0) = \varphi_2(x^0)$, and this value be finite. If the function φ_2 is directionally Lipschitz at x^0 and the condition (28) is fulfilled, then

$$D^-(\varphi_1 \vee \varphi_2)(x^0) \subset \bigcup_{\substack{\lambda_1\geqslant 0,\lambda_2\geqslant 0\\ \lambda_1+\lambda_2=1}} [\lambda_1 \circ D^-\varphi_1(x^0) + \lambda_2 \circ D^-\varphi_2(x^0)]. \tag{33}$$

If, in addition, the sets $E(\varphi_1)$ and $E(\varphi_2)$ are regular at the point $(x^0, \varphi_1(x^0))$, then the function $\varphi_1 \vee \varphi_2$ is lower regular at x^0 and equality holds in (33).

In (33), by definition,

$$\lambda_i \circ D^-\varphi_i(x^0) = \begin{cases} \lambda_i D^-\varphi_i(x^0), & \text{if } \lambda_i > 0, \\ D_s^-\varphi_i(x^0), & \text{if } \lambda_i = 0. \end{cases}$$

COROLLARY. If the functions $\varphi_i : X \to R$, i = 1, 2,...,m, satisfy the Lipschitz condition in the neighborhood of a point x^0, then

$$D^-\left(\bigvee_{i=1}^m \varphi_i\right)(x^0) \subset \left[\sum_{i=1}^m \lambda_i D^-\varphi_i(x^0) : \lambda_i \geqslant 0,\ \lambda_i\left(\varphi_i(x^0) - \bigvee_{j=1}^m \varphi_j(x^0)\right) = 0,\ \ i = 1, 2, \ldots, m;\ \sum_{i=1}^m \lambda_i = 1\right]. \tag{34}$$

If, moreover, the functions φ_i, $i = 1, 2, \ldots, m$, are lower regular at x^0, then the function $\bigvee_{i=1}^{m} \varphi_i$ is also lower regular at this point and equality holds in (34).

THEOREM 12. Let a function $\varphi: X \to Z$ be continuous in the neighborhood of a point x^0 and a function $g: Z \to \bar{R}$ be finite at the point $z^0 = \varphi(x^0)$ and lower semicontinuous in its neighborhood. If g is directionally Lipschitz at z^0 and there exists an element $(\bar{x}, \bar{z}) \in T_C(x^0, z^0 | G(\varphi))$ such that $\bar{z} \in \operatorname{int} \operatorname{dom} g^{\uparrow}(z^0; \cdot)$, then

$$D^-(g \circ \varphi)(x^0) \subset \{x^* \in X^* : (x^*, -z^*) \in K(x^0, z^0 | G(\varphi)) \quad \text{for a certain element} \quad z^* \in D^- g(z^0)\}.$$

In addition, if the function φ satisfies the Lipschitz condition in the neighborhood of x^0, then

$$D^-(g \circ \varphi)(x^0) \subset \bigcup \left[D_{z^*}\varphi(x^0) : z^* \in D^- g(z^0)\right]. \tag{35}$$

If, moreover, $\dim Z < \infty$, the function g is lower regular at z^0, and for each $z^* \in D^- g(z^0)$ the function $x \to \langle z^*, \varphi(x) \rangle$ is lower regular at x^0, then the function $g \circ \varphi$ is also lower regular at x^0 and equality holds in (35).

As a consequence of Theorem 12, we can deduce a generalization of the classical mean-value theorem.

THEOREM 13. If a function $\varphi: X \to R$ satisfies the Lipschitz condition in a domain that contains the segment $[x_1, x_2]$, then there exist points $x^0 \in]x_1, x_2[$ and $x^* \in D^0\varphi(x^0)$ such that

$$\varphi(x_2) - \varphi(x_1) = \langle x^*, x_2 - x_1 \rangle.$$

By virtue of the minimality of Clarke's generalized gradient in the class of weakly* closed convex generalized derivatives, for which the analogue of the mean-value theorem is valid, established by Lebourg [12], the following result, refining Corollary 2 of Theorem 7, follows from Theorem 13.

COROLLARY. If the function φ satisfies the Lipschitz condition in the neighborhood of a point x^0, then

$$\partial_C \varphi(x^0) = \operatorname{cl} \operatorname{co} D^0 \varphi(x^0).$$

By virtue of this corollary, the results of this section generalize and strengthen the corresponding statements for Clarke's generalized gradient [10, 11, 14-17].

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